

AMC 12 SOLUTION ARCHIVE

2024

12A and 12B full explanations

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2024 AMC 12A Solutions

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1. What is the value of $9901 \cdot 101 - 99 \cdot 10101$?

☒ A 2

☐ B 20

☐ C 21

☐ D 200

☐ E 2020

Solution:

Directly, $9901 \cdot 101 = 990100 + 9901 = 1000001$ and $99 \cdot 10101 = 1010100 - 10101 = 999999$. Their difference is $1000001 - 999999 = 2$. Thus, the correct answer is **A**.

2. A model used to estimate the time it will take to hike to the top of a mountain on a trail is of the form $T = aL + bG$, where a and b are constants, T is the time in minutes, L is the length of the trail in miles, and G is the altitude gain in feet. The model estimates that it will take 69 minutes to hike to the top if a trail is 1.5 miles long and ascends 800 feet, as well as if a trail is 1.2 miles long and ascends 1100 feet.

How many minutes does the model estimate it will take to hike to the top if the trail is 4.2 miles long and ascends 4000 feet?

A 240

B 246

C 252

D 258

E 264

Solution:

From $1.5a + 800b = 1.2a + 1100b$ we get $0.3a = 300b$, so $a = 1000b$. Then $69 = 1.5a + 800b = 1500b + 800b = 2300b$, giving $b = 0.03$ and $a = 30$. For $L = 4.2$, $G = 4000$, $T = 30(4.2) + 0.03(4000) = 126 + 120 = 246$. Thus, the correct answer is **B**.

3. The number 2024 is written as the sum of not necessarily distinct two-digit numbers. What is the least number of two-digit numbers needed to write this sum?

- ☐ A 20
- ☒ B 21
- ☐ C 22
- ☐ D 23
- ☐ E 24

Solution:

Each number is at most 99, so k numbers sum to at most $99k$. Since $99 \cdot 20 = 1980 < 2024$, at least 21 numbers are required. With 21, we can use twenty 99s and one 44: $20 \cdot 99 + 44 = 1980 + 44 = 2024$. Thus, the correct answer is **B**.

4. What is the least value of n such that $n!$ is a multiple of 2024?

- ☐ A 11
- ☐ B 21
- ☐ C 22
- ☒ D 23
- ☐ E 253

Solution:

Factoring, $2024 = 2^3 \cdot 11 \cdot 23$. The factorial $n!$ contains the prime 23 only when $n \geq 23$. At $n = 23$, the product $23!$ already includes 23, 11, and plenty of factors of 2, so $23!$ is a multiple of 2024. Thus, the correct answer is **D**.

5. A data set containing 20 numbers, some of which are 6, has mean 45. When all the 6s are removed, the data set has mean 66. How many 6s were in the original data set?

- ☐ A 4
- ☐ B 5
- ☐ C 6
- ☒ D 7
- ☐ E 8

Solution:

The full set sums to $20 \cdot 45 = 900$. Removing k sixes leaves $20 - k$ numbers summing to $900 - 6k$, with mean 66, so $900 - 6k = 66(20 - k) = 1320 - 66k$. Then $60k = 420$, giving $k = 7$. Thus, the correct answer is **D**.

6. The product of three integers is 60. What is the least possible positive sum of the three integers?

- A 2
- B 3**
- C 5
- D 6
- E 13

Solution:

To obtain a small positive sum, use two negative integers $-p$, $-q$ and one positive integer r , where $pqr = 60$. Up to order, the positive factor triples of 60 are

$(1, 1, 60), (1, 2, 30),$
 $(1, 3, 20), (1, 4, 15),$
 $(1, 5, 12), (1, 6, 10),$
 $(2, 2, 15), (2, 3, 10),$
 $(2, 5, 6), (3, 4, 5).$

A positive value of $r - p - q$ is smallest when the largest factor is chosen as r . The positive values from the list are 58, 27, 16, 10, 6, 3, 11, 5; the last two triples give negative values. Thus the least positive sum is $10 - 1 - 6 = 3$. (Three positive integers have sum at least 3, and three negative integers cannot have positive product.) Thus, the correct answer is **B**.

7. In $\triangle ABC$, $\angle ABC = 90^\circ$ and $BA = BC = \sqrt{2}$. Points $P_1, P_2, \dots, P_{2024}$ lie on hypotenuse AC so that $AP_1 = P_1P_2 = P_2P_3 = \dots = P_{2023}P_{2024} = P_{2024}C$.

What is the length of the vector sum

$$\begin{aligned} & \vec{BP}_1 + \vec{BP}_2 + \vec{BP}_3 \\ & + \dots + \vec{BP}_{2024}? \end{aligned}$$

- ☐ A 1011
- ☐ B 1012
- ☐ C 2023
- ☒ D 2024
- ☐ E 2025

Solution:

The points P_k are symmetric about the midpoint M of AC , so pairing P_k with its mirror gives $\vec{BP}_k + \vec{BP}_{2025-k} = 2\vec{BM}$. Hence the whole sum is $2024\vec{BM}$. In a right triangle the median to the hypotenuse has length half the hypotenuse; here $AC = 2$, so $BM = 1$. The length of the sum is $2024 \cdot 1 = 2024$. Thus, the correct answer is **D**.

8. How many angles θ with $0 \leq \theta \leq 2\pi$ satisfy $\log(\sin(3\theta)) + \log(\cos(2\theta)) = 0$?

A 0

B 1

C 2

D 3

E 4

Solution:

The equation means $\sin(3\theta) \cos(2\theta) = 1$ with both factors positive (for the logs to be defined). Since $\sin(3\theta) \leq 1$ and $\cos(2\theta) \leq 1$, their product is 1 only if $\sin(3\theta) = 1$ and $\cos(2\theta) = 1$ simultaneously. But $\cos(2\theta) = 1$ forces $\theta \in \{0, \pi, 2\pi\}$, where $\sin(3\theta) = 0 \neq 1$. No angle works. Thus, the correct answer is **A**.

9. Let M be the greatest integer such that both $M + 1213$ and $M + 3773$ are perfect squares. What is the units digit of M ?

A 1

B 2

C 3

D 6

E 8

Solution:

Write $M + 1213 = a^2$ and $M + 3773 = b^2$, so $b^2 - a^2 = 2560$, i.e. $(b - a)(b + a) = 2560$. Both factors have the same parity, hence both even. To maximize a (and thus M), minimize $b - a$: take $b - a = 2$, $b + a = 1280$, so $a = 639$. Then $M = 639^2 - 1213 = 408321 - 1213 = 407108$, whose units digit is 8. Thus, the correct answer is **E**.

10. Let α be the radian measure of the smallest angle in a 3-4-5 right triangle. Let β be the radian measure of the smallest angle in a 7-24-25 right triangle. In terms of α , what is β ?

- A $\frac{\alpha}{3}$
- B $\alpha - \frac{\pi}{8}$
- C $\frac{\pi}{2} - 2\alpha$**
- D $\frac{\alpha}{2}$
- E $\pi - 4\alpha$

Solution:

The smallest angle of the 3-4-5 triangle has $\tan \alpha = \frac{3}{4}$. Then

$$\begin{aligned}\tan 2\alpha &= \frac{2 \cdot \frac{3}{4}}{1 - \frac{9}{16}} \\ &= \frac{\frac{3}{2}}{\frac{7}{16}} = \frac{24}{7}.\end{aligned}$$

The smallest angle of the 7-24-25 triangle has $\tan \beta = \frac{7}{24} = \cot 2\alpha = \tan\left(\frac{\pi}{2} - 2\alpha\right)$. Hence $\beta = \frac{\pi}{2} - 2\alpha$. Thus, the correct answer is **C**.

11. There are exactly K positive integers b with $5 \leq b \leq 2024$ such that the base- b integer 2024_b is divisible by 16 (where 16 is in base ten). What is the sum of the digits of K ?

A 16

B 17

C 18

D 20

E 21

Solution:

Here $2024_b = 2b^3 + 2b + 4 = 2(b^3 + b + 2)$, so $16 \mid 2024_b$ exactly when $8 \mid b^3 + b + 2$. Checking residues mod 8, $b^3 + b + 2 \equiv 0$ precisely for $b \equiv 3, 6, 7 \pmod{8}$.

Counting b in $[5, 2024]$: residue 3 gives $11, \dots, 2019$ (252 values), residue 6 gives $6, \dots, 2022$ (253 values), and residue 7 gives $7, \dots, 2023$ (253 values). So $K = 252 + 253 + 253 = 758$, and its digit sum is $7 + 5 + 8 = 20$.

Thus, the correct answer is **D**.

12. The first three terms of a geometric sequence are the integers a , 720, and b , where $a < 720 < b$. What is the sum of the digits of the least possible value of b ?

- A 9
- B 12
- C 16
- D 18
- E 21**

Solution:

Since the terms are geometric, $720^2 = ab$, so $ab = 518400 = 2^8 \cdot 3^4 \cdot 5^2$. Because $b = \frac{518400}{a}$, minimizing b means finding the smallest divisor of 518400 greater than 720. There is no divisor strictly between 720 and 768 : if its exponent of 5 is 2, 1, or 0, then after dividing by 25, 5, or 1, respectively, a number of the form $2^i 3^j$ would have to lie in $(28.8, 30.72)$, $(144, 153.6)$, or $(720, 768)$; the allowed powers $i \leq 8, j \leq 4$ give none. Since $768 = 2^8 \cdot 3$ is a divisor, it is the least possible b . Its paired divisor is $a = \frac{518400}{768} = 675$, and the digit sum of b is $7 + 6 + 8 = 21$. Thus, the correct answer is **E**.

13. The graph of $y = e^{x+1} + e^{-x} - 2$ has an axis of symmetry. What is the reflection of the point $(-1, \frac{1}{2})$ over this axis?

A $(-1, -\frac{3}{2})$

B $(-1, 0)$

C $(-1, \frac{1}{2})$

D $(0, \frac{1}{2})$

E $(3, \frac{1}{2})$

Solution:

The curve $y = e^{x+1} + e^{-x} - 2$ is symmetric about the vertical line through its minimum. Setting the derivative $e^{x+1} - e^{-x} = 0$ gives $x + 1 = -x$, so $x = -\frac{1}{2}$. Reflecting $(-1, \frac{1}{2})$ across $x = -\frac{1}{2}$ keeps the y -coordinate and sends $x = -1$ to $x = 0$. The image is $(0, \frac{1}{2})$. Thus, the correct answer is **D**.

14. The numbers, in order, of each row and the numbers, in order, of each column of a 5×5 array of integers form an arithmetic progression of length 5. The numbers in positions $(5, 5)$, $(2, 4)$, $(4, 3)$, and $(3, 1)$ are 0, 48, 16, and 12, respectively. What number is in position $(1, 2)$?

$$\begin{bmatrix} \cdot & ? & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 48 & \cdot \\ 12 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 16 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 \end{bmatrix}$$

A 19

B 24

C 29

D 34

E 39

Solution:

A grid whose rows and columns are all arithmetic has entries of the bilinear form $a(i, j) = \alpha + \beta i + \gamma j + \delta ij$. The four givens yield

$$\alpha + 5\beta + 5\gamma + 25\delta = 0,$$

$$\alpha + 2\beta + 4\gamma + 8\delta = 48,$$

$$\alpha + 4\beta + 3\gamma + 12\delta = 16,$$

$$\alpha + 3\beta + \gamma + 3\delta = 12.$$

Solving gives $\delta = -5$, $\beta = 5$, $\gamma = 22$, $\alpha = -10$. Then $a(1, 2) = \alpha + \beta + 2\gamma + 2\delta = -10 + 5 + 44 - 10 = 29$.

Thus, the correct answer is **C**.

15. The roots of $x^3 + 2x^2 - x + 3$ are p , q , and r . What is the value of

$$(p^2 + 4)(q^2 + 4)(r^2 + 4)?$$

A 64

B 75

C 100

D 125

E 144

Solution:

Since $P(x) = (x - p)(x - q)(x - r)$, grouping $p^2 + 4 = (p - 2i)(p + 2i)$ over all roots gives

$$\prod (p^2 + 4) = P(2i) P(-2i).$$

Compute $P(2i) = -8i - 8 - 2i + 3 = -5 - 10i$ and $P(-2i) = 8i - 8 + 2i + 3 = -5 + 10i$. Their product is $(-5)^2 + 10^2 = 25 + 100 = 125$. Thus, the correct answer is **D**.

16. A set of 12 tokens — 3 red, 2 white, 1 blue, and 6 black — is to be distributed at random to 3 game players, 4 tokens per player. The probability that some player gets all the red tokens, another gets all the white tokens, and the remaining player gets the blue token can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n$?

A 387

B 388

C 389

D 390

E 391

Solution:

Treat all tokens as distinct; the total number of ways to deal 4 to each player is $\binom{12}{4,4,4} = 34650$. For the favorable event, choose which player gets the reds, whites, and blue in $3! = 6$ ways. The red player needs 1 more token, the white player 2 more, and the blue player 3 more, all black; the 6 black tokens split as 1, 2, 3 in $\frac{6!}{1!2!3!} = 60$ ways. So the probability is $\frac{6 \cdot 60}{34650} = \frac{360}{34650} = \frac{4}{385}$. Then $m + n = 4 + 385 = 389$. Thus, the correct answer is **C**.

17. Integers a , b , and c satisfy $ab + c = 100$, $bc + a = 87$, and $ca + b = 60$. What is $ab + bc + ca$?

A 212

B 247

C 258

D 276

E 284

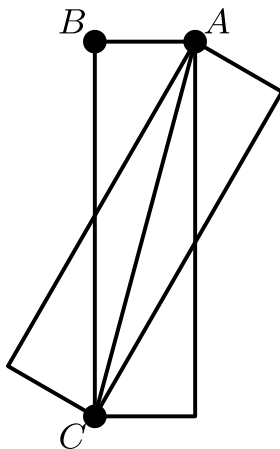
Solution:

Subtracting the second equation from the first gives $(a - c)(b - 1) = 13$. Hence

$$(a - c, b - 1) \in \{(1, 13), (13, 1), (-1, -13), (-13, -1)\}.$$

The corresponding values of b are 14, 2, -12 , 0. Substituting $a = c + (a - c)$ into $ab + c = 100$ eliminates the first two cases because they would require $15c = 86$ or $3c = 74$. The case $b = -12$ gives $c = -8$ and $a = -9$, which satisfies all three equations. The case $b = 0$ gives $(a, c) = (87, 100)$, which fails $ca + b = 60$. Thus the unique integer solution is $(-9, -12, -8)$, and $ab + bc + ca = 108 + 96 + 72 = 276$. Thus, the correct answer is **D**.

18. On top of a rectangular card with sides of length 1 and $2 + \sqrt{3}$, an identical card is placed so that two of their diagonals line up, as shown (AC , in this case).



Continue the process, adding a third card to the second, and so on, lining up successive diagonals after rotating clockwise. In total, how many cards must be used until a vertex of a new card lands exactly on the vertex labeled B in the figure?

- ☒ A 6
- ☐ B 8
- ☐ C 10
- ☐ D 12
- ☐ E No new vertex will land on B .

Solution:

A diagonal makes an angle θ with a long side, where $\tan \theta = \frac{1}{2 + \sqrt{3}} = 2 - \sqrt{3} = \tan 15^\circ$. Thus the acute angle between the two diagonals of a card is $2\theta = 30^\circ$.

Each new card shares one diagonal with the previous card, and its other diagonal is the next line obtained by turning 30° clockwise. All these equal diagonals have the same midpoint and are diameters of one common circle. The line through the original card's other diagonal, which contains B , is 30° counterclockwise from AC . As an unoriented line, this is the same as 150° clockwise from AC . Five additions advance the unused diagonal by $5 \cdot 30^\circ = 150^\circ$, so the sixth card is the first new card with a vertex at B .

Thus, the correct answer is **A**.

19. Cyclic quadrilateral $ABCD$ has lengths $BC = CD = 3$ and $DA = 5$ with $\angle CDA = 120^\circ$. What is the length of the shorter diagonal of $ABCD$?

☐ A $\frac{31}{7}$

☐ B $\frac{33}{7}$

☐ C 5

☒ D $\frac{39}{7}$

☐ E $\frac{41}{7}$

Solution:

In $\triangle ACD$, the law of cosines gives $AC^2 = 9 + 25 - 2(15) \cos 120^\circ = 34 + 15 = 49$, so $AC = 7$.

Since $ABCD$ is cyclic, $\angle ABC = 180^\circ - 120^\circ = 60^\circ$. In $\triangle ABC$ with $BC = 3$ and $AC = 7$, the law of cosines gives $49 = AB^2 + 9 - 3AB$, so $AB = 8$. By Ptolemy, $AC \cdot BD = AB \cdot CD + BC \cdot DA = 8 \cdot 3 + 3 \cdot 5 = 39$, hence $BD = \frac{39}{7}$. This is shorter than $AC = 7$.

Thus, the correct answer is **D**.

20. Points P and Q are chosen uniformly and independently at random on sides $\overline{AB}$ and $\overline{AC}$, respectively, of equilateral triangle $\triangle ABC$. Which of the following intervals contains the probability that the area of $\triangle APQ$ is less than half the area of $\triangle ABC$?

A $\left[\frac{3}{8}, \frac{1}{2}\right]$

B $\left(\frac{1}{2}, \frac{2}{3}\right]$

C $\left(\frac{2}{3}, \frac{3}{4}\right]$

D $\left(\frac{3}{4}, \frac{7}{8}\right]$

E $\left(\frac{7}{8}, 1\right]$

Solution:

With $x = \frac{AP}{AB}$ and $y = \frac{AQ}{AC}$ uniform on $[0, 1]$, the area ratio $\frac{[APQ]}{[ABC]} = xy$. The complementary event $xy \geq \frac{1}{2}$ requires $x \geq \frac{1}{2}$ and $y \in [\frac{1}{2x}, 1]$, with probability

$$\begin{aligned} & \int_{\frac{1}{2}}^1 \left(1 - \frac{1}{2x}\right) dx \\ &= \frac{1}{2} - \frac{\ln 2}{2} \approx 0.153. \end{aligned}$$

Therefore $P(xy < \frac{1}{2}) \approx 1 - 0.153 = 0.847$, which lies in $\left(\frac{3}{4}, \frac{7}{8}\right]$. Thus, the correct answer is **D**.

21. Suppose that $a_1 = 2$ and the sequence (a_n) satisfies the recurrence relation

$$\frac{a_n - 1}{n - 1} = \frac{a_{n-1} + 1}{(n - 1) + 1}$$

for all $n \geq 2$. What is the greatest integer less than or equal to

$$\sum_{n=1}^{100} a_n^2?$$

A 338,550

B 338,551

C 338,552

D 338,553

E 338,554

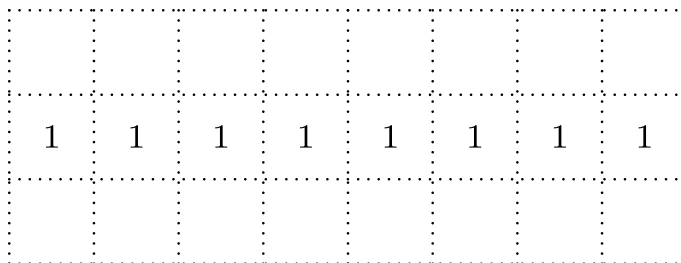
Solution:

The recurrence rearranges to $a_n = 1 + \frac{n-1}{n}(a_{n-1} + 1)$. Computing early terms $2, \frac{5}{2}, \frac{10}{3}, \frac{17}{4}, \dots$ suggests $a_n = n + \frac{1}{n}$. This follows by induction: substituting $a_{n-1} = n - 1 + \frac{1}{n-1}$ into the recurrence gives $a_n = 1 + \frac{n-1}{n}(n + \frac{1}{n-1}) = n + \frac{1}{n}$. Then $a_n^2 = n^2 + 2 + \frac{1}{n^2}$, so

$$\begin{aligned} \sum_{n=1}^{100} a_n^2 &= \sum_{n=1}^{100} n^2 + 200 \\ &\quad + \sum_{n=1}^{100} \frac{1}{n^2} \\ &= 338350 + 200 + S, \end{aligned}$$

where $S > 1$ and $S < 1 + \int_1^\infty x^{-2} dx = 2$. Hence the sum is between **338551** and **338552**, and its floor is **338551**. Thus, the correct answer is **B**.

22. The figure below shows a dotted grid 8 cells wide and 3 cells tall consisting of $1'' \times 1''$ squares. Carl places 1-inch toothpicks along some of the sides of the squares to create a closed loop that does not intersect itself. The numbers in the cells indicate the number of sides of that square that are to be covered by toothpicks, and any number of toothpicks are allowed if no number is written. In how many ways can Carl place the toothpicks?



- A 130
- B 144
- C 146**
- D 162
- E 196

Solution:

Each middle-row cell must touch exactly one toothpick. First consider loops that pass from one side of the middle strip to the other. The loop can span all 8 columns, the first 7, the last 7, or the middle 6; a narrower span would leave an outer middle cell untouched.

Once the two ends are fixed, each interior middle cell independently has its one toothpick on its top or bottom side, and the rest of the non-self-intersecting loop is forced. The four cases therefore contribute 2^6 , 2^5 , 2^5 , and 2^4 loops. There are also exactly two loops that do not cross the middle strip: the horizontal rectangle running entirely along the top or entirely along the bottom. Hence the total is $2^6 + 2^5 + 2^5 + 2^4 + 2 = 64 + 32 + 32 + 16 + 2 = 146$.

Thus, the correct answer is **C**.

23. What is the value of

$$\begin{aligned} & \tan^2 \frac{\pi}{16} \cdot \tan^2 \frac{3\pi}{16} \\ & + \tan^2 \frac{\pi}{16} \cdot \tan^2 \frac{5\pi}{16} \\ & + \tan^2 \frac{3\pi}{16} \cdot \tan^2 \frac{7\pi}{16} \\ & + \tan^2 \frac{5\pi}{16} \cdot \tan^2 \frac{7\pi}{16} ? \end{aligned}$$

A 28

B 68

C 70

D 72

E 84

Solution:

With $a = \tan^2 \frac{\pi}{16}$, $b = \tan^2 \frac{3\pi}{16}$, $c = \tan^2 \frac{5\pi}{16}$, $d = \tan^2 \frac{7\pi}{16}$, the expression is $ab + ac + bd + cd = (a + d)(b + c)$.

Since $\frac{7\pi}{16} = \frac{\pi}{2} - \frac{\pi}{16}$, we have $d = \cot^2 \frac{\pi}{16}$, so $a + d = \tan^2 \frac{\pi}{16} + \cot^2 \frac{\pi}{16} = \frac{4}{\sin^2(\frac{\pi}{8})} - 2 = 14 + 8\sqrt{2}$. Likewise $b + c = \frac{4}{\sin^2(\frac{3\pi}{8})} - 2 = 14 - 8\sqrt{2}$. Their product is $14^2 - (8\sqrt{2})^2 = 196 - 128 = 68$.

Thus, the correct answer is **B**.

24. A *disphenoid* is a tetrahedron whose triangular faces are congruent to one another. What is the least total surface area of a disphenoid whose faces are scalene triangles with integer side lengths?

A $\sqrt{3}$

B $3\sqrt{15}$

C 15

D $15\sqrt{7}$

E $24\sqrt{6}$

Solution:

A disphenoid exists (as the tetrahedron formed by the face-plane midpoints of a box) exactly when the common face triangle is acute, and its total surface area is 4 times one face's area. Write the integer side lengths as $u < v < w$. If $u \leq 3$, then $w \geq v + 1$ and $u^2 + v^2 \leq (v + 1)^2$ (with equality only for $(u, v, w) = (3, 4, 5)$), so the triangle is not acute. Thus $u \geq 4$.

The triangle $(4, 5, 6)$ is acute because $4^2 + 5^2 > 6^2$. It also has the least possible area. The largest angle of any acute triangle is at least 60° , so a candidate with $(u, v) \neq (4, 5)$ has area at least $\frac{1}{2}uv \sin 60^\circ \geq \frac{1}{2}(4)(6)\frac{\sqrt{3}}{2} = 6\sqrt{3}$, which is greater than $\frac{15\sqrt{7}}{4}$, the area of $(4, 5, 6)$.

By Heron's formula with $s = \frac{15}{2}$, that area is $\sqrt{\frac{15}{2} \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2}} = \frac{15\sqrt{7}}{4}$. The total surface area is $4 \cdot \frac{15\sqrt{7}}{4} = 15\sqrt{7}$.

Thus, the correct answer is **D**.

25. A graph is *symmetric* about a line if the graph remains unchanged after reflection in that line. For how many quadruples of integers (a, b, c, d) , where $|a|, |b|, |c|, |d| \leq 5$ and c and d are not both 0, is the graph of

$$y = \frac{ax + b}{cx + d}$$

symmetric about the line $y = x$?

A 1282

B 1292

C 1310

D 1320

E 1330

Solution:

Reflecting the graph of $y = f(x)$ over $y = x$ produces the graph of its inverse, so the graph is symmetric about $y = x$ exactly when f equals its own inverse. For $f(x) = \frac{ax+b}{cx+d}$ this happens in two ways: when $a + d = 0$ with $ad - bc \neq 0$ (a genuine involution, including the slope -1 lines when $c = 0$), or when f is the identity $y = x$ ($b = c = 0, a = d \neq 0$).

For $a + d = 0$, set $d = -a$; the determinant $-a^2 - bc$ must be nonzero, so we need $a^2 + bc \neq 0$, together with $(c, d) \neq (0, 0)$. When $a = 0$, both b and c must be nonzero, giving $10^2 = 100$ choices. For each nonzero a , start with $11^2 = 121$ choices of (b, c) . If $|a| = 1, 3, 4$, or 5 , exactly 2 pairs satisfy $bc = -a^2$; if $|a| = 2$, exactly 6 pairs do. Thus the genuine involutions number

$$\begin{aligned} &100 + 8(121 - 2) + 2(121 - 6) \\ &= 1282. \end{aligned}$$

The identity case adds 10 more ($a = d \in \{\pm 1, \dots, \pm 5\}$), for a total of $1282 + 10 = 1292$.

Thus, the correct answer is **B**.

Problems: <https://live.poshenloh.com/past-contests/amc12/2024A>



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1. In a long line of people arranged left to right, the 1013th person from the left is also the 1010th person from the right. How many people are in the line?

A 2021

B 2022

C 2023

D 2024

E 2025

Solution:

The target person has $1013 - 1 = 1012$ people to the left and $1010 - 1 = 1009$ people to the right. Including the person themselves, the line has $1012 + 1009 + 1 = 2022$ people.

Thus, the correct answer is **B**.

2. What is $10! - 7! \cdot 6!$?

A -120

B 0

C 120

D 600

E 720

Solution:

Note that $6! = 720 = 8 \cdot 9 \cdot 10$. Therefore

$$7! \cdot 6! = 7! \cdot (8 \cdot 9 \cdot 10) = 10!.$$

So $10! - 7! \cdot 6! = 10! - 10! = 0$.

Thus, the correct answer is **B**.

3. For how many integer values of x is $|2x| \leq 7\pi$?

A 16

B 17

C 19

D 20

E 21

Solution:

The inequality $|2x| \leq 7\pi$ is equivalent to $|x| \leq \frac{7\pi}{2} \approx 10.99$. The integers satisfying this run from -10 to 10 , which is $10 + 10 + 1 = 21$ values.

Thus, the correct answer is **E**.

4. Balls numbered $1, 2, 3, \dots$ are deposited in 5 bins, labeled A, B, C, D , and E , using the following procedure. Ball 1 is deposited in bin A , and balls 2 and 3 are deposited in B . The next three balls are deposited in bin C , the next 4 in bin D , and so on, cycling back to bin A after balls are deposited in bin E . (For example, 22, 23, $\dots$, 28 are deposited in bin B at step 7 of this process.) In which bin is ball 2024 deposited?

☐ A A

☐ B B

☐ C C

☒ D D

☐ E E

Solution:

Step k deposits k balls, so after step k a total of $\frac{k(k+1)}{2}$ balls have been placed.

Since $\frac{63 \cdot 64}{2} = 2016$ and $\frac{64 \cdot 65}{2} = 2080$, ball 2024 falls in step 64.

The steps cycle through the bins A, B, C, D, E , so step k uses position $(k - 1) \bmod 5$. Here $(64 - 1) \bmod 5 = 63 \bmod 5 = 3$, which is the fourth bin, D .

Thus, the correct answer is **D**.

5. In the following expression, Melanie changed some of the plus signs to minus signs:

$$1 + 3 + 5 + 7 + \cdots + 97 + 99$$

When the new expression was evaluated, it was negative. What is the least number of plus signs that Melanie could have changed to minus signs?

- ☐ A 14
- ☒ B 15
- ☐ C 16
- ☐ D 17
- ☐ E 18

Solution:

The original expression sums the first 50 odd numbers, giving $50^2 = 2500$. Changing a term of value v from $+$ to $-$ decreases the total by $2v$, so to make the result negative the flipped terms must total more than $\frac{2500}{2} = 1250$.

To use as few terms as possible, flip the largest odd numbers 99, 97, 95, ... The largest k of them sum to $k(100 - k)$. With $k = 14$ this is $14 \cdot 86 = 1204 \leq 1250$, but with $k = 15$ it is $15 \cdot 85 = 1275 > 1250$. So 15 sign changes suffice and 14 do not.

Thus, the correct answer is **B**.

6. The national debt of the United States is on track to reach $5 \cdot 10^{13}$ dollars by 2033. How many digits does this number of dollars have when written as a numeral in base 5? (The approximation of $\log_{10} 5$ as 0.7 is sufficient for this problem.)

A 18

B 20

C 22

D 24

E 26

Solution:

The number of digits of N in base 5 is $\lfloor \log_5 N \rfloor + 1$. With $N = 5 \cdot 10^{13}$,

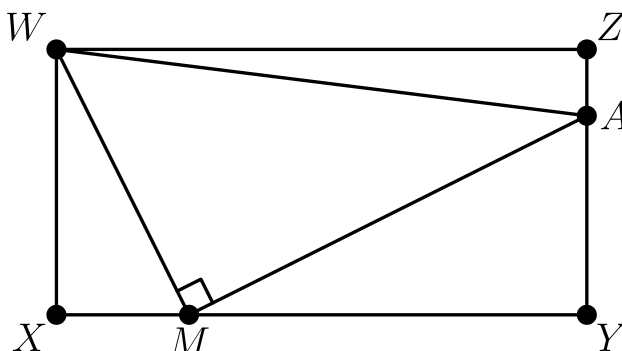
$$\log_{10} N = 13 + \log_{10} 5 = 13.7.$$

Converting bases, $\log_5 N = \frac{13.7}{\log_{10} 5} = \frac{13.7}{0.7} = 19.57 \dots$

Thus the number of digits is $\lfloor 19.57 \rfloor + 1 = 19 + 1 = 20$.

Thus, the correct answer is **B**.

7. In the figure below $WXYZ$ is a rectangle with $WX = 4$ and $WZ = 8$. Point M lies on $\overline{XY}$, point A lies on $\overline{YZ}$, and $\angle WMA$ is a right angle. The areas of $\triangle WXM$ and $\triangle WAZ$ are equal. What is the area of $\triangle WMA$?



A 13

B 14

C 15

D 16

E 17

Solution:

Set $X = (0, 0)$, $W = (0, 4)$, $Y = (8, 0)$, $Z = (8, 4)$, with $M = (m, 0)$ on $\overline{XY}$ and $A = (8, a)$ on $\overline{YZ}$.

Since $\angle WMA = 90^\circ$, $\overrightarrow{MW} \cdot \overrightarrow{MA} = (-m)(8 - m) + 4a = 0$, so $4a = m(8 - m)$. The areas give $[\triangle WXM] = \frac{1}{2} \cdot 4 \cdot m = 2m$ and $[\triangle WAZ] = \frac{1}{2} \cdot 8 \cdot (4 - a) = 4(4 - a)$. Setting these equal yields $m = 8 - 2a$.

Substituting $a = \frac{8-m}{2}$ into $4a = m(8 - m)$ gives $2(8 - m) = m(8 - m)$, so $m = 2$ and $a = 3$. The other algebraic root, $m = 8$, gives $M = A = Y$ and no defined angle $\angle WMA$, so it is invalid. Then with $W = (0, 4)$, $M = (2, 0)$, $A = (8, 3)$,

$$\begin{aligned} [\triangle WMA] &= \frac{1}{2} |2(3 - 4) + 8(4 - 0)| \\ &= \frac{1}{2}(30) = 15. \end{aligned}$$

Thus, the correct answer is **C**.

8. What value of x satisfies

$$\frac{\log_2 x \cdot \log_3 x}{\log_2 x + \log_3 x} = 2?$$

A 25

B 32

C 36

D 42

E 48

Solution:

Dividing top and bottom by $\log_2 x \cdot \log_3 x$, the left side becomes

$$\begin{aligned} & \frac{1}{\frac{1}{\log_2 x} + \frac{1}{\log_3 x}} \\ &= \frac{1}{\log_x 2 + \log_x 3} \\ &= \frac{1}{\log_x 6}. \end{aligned}$$

So $\frac{1}{\log_x 6} = 2$, meaning $\log_x 6 = \frac{1}{2}$, i.e. $x^{\frac{1}{2}} = 6$.

Therefore $x = 36$.

Thus, the correct answer is **C**.

9. A dartboard is the region B in the coordinate plane consisting of points (x, y) such that $|x| + |y| \leq 8$. A target T is the region where $(x^2 + y^2 - 25)^2 \leq 49$. A dart is thrown and lands at a random point in B . The probability that the dart lands in T can be expressed as $\frac{m}{n} \cdot \pi$, where m and n are relatively prime positive integers. What is $m + n$?

A 39

B 71

C 73

D 75

E 135

Solution:

The dartboard $|x| + |y| \leq 8$ is a square with diagonals 16, so its area is $\frac{1}{2} \cdot 16 \cdot 16 = 128$. The target condition $(x^2 + y^2 - 25)^2 \leq 49$ means $-7 \leq x^2 + y^2 - 25 \leq 7$, i.e. $18 \leq x^2 + y^2 \leq 32$, an annulus of area $\pi(32 - 18) = 14\pi$.

The distance from the origin to a side of the square (for instance $x + y = 8$) is $\frac{8}{\sqrt{2}} = \sqrt{32}$, exactly the annulus's outer radius. So the annulus is tangent to the square and lies entirely within B . The probability is $\frac{14\pi}{128} = \frac{7}{64}\pi$, giving $m + n = 7 + 64 = 71$.

Thus, the correct answer is **B**.

10. A list of 9 real numbers consists of 1, 2.2, 3.2, 5.2, 6.2, and 7, as well as x, y, z with $x \leq y \leq z$. The range of the list is 7, and the mean and median are both positive integers. How many ordered triples (x, y, z) are possible?

- A 1
- B 2
- C 3**
- D 4
- E infinitely many

Solution:

The six fixed numbers sum to 24.8 and span $[1, 7]$. There are three possible arrangements of the overall extremes.

If the extremes are 0 and 7, then $x = 0$ and $z \leq 7$. The only possible integer means are 3 and 4, requiring $y + z = 2.2$ or 11.2. The first makes the median 2.2. In the second, an integer median forces $y = 5$, hence $z = 6.2$. This gives $(0, 5, 6.2)$.

If the extremes are 1 and 8, then $z = 8$ and the integer mean forces $x + y = 3.2$ or 12.2. The first makes the median 3.2; the second has an integer median only for $x = 6$, $y = 6.2$. This gives $(6, 6.2, 8)$.

Finally, if both extremes are new, write $x = t$, $z = t + 7$ with $0 < t < 1$. The mean must be 4, so $y = 4.2 - 2t$. The median is the fourth number among 1, 2.2, 3.2, 5.2, 6.2, 7, y ; it is an integer only when $y = 4$, giving $t = 0.1$. Thus the third triple is $(0.1, 4, 7.1)$, and there are exactly 3 in all.

Thus, the correct answer is **C**.

11. Let $x_n = \sin^2(n^\circ)$. What is the mean of $x_1, x_2, x_3, \dots, x_{90}$?

- A $\frac{11}{45}$
- B $\frac{22}{45}$
- C $\frac{89}{180}$
- D $\frac{1}{2}$
- E $\frac{91}{180}$

Solution:

$$\text{Using } \sin^2 \theta = \frac{1 - \cos 2\theta}{2},$$

$$\begin{aligned} \sum_{n=1}^{90} \sin^2(n^\circ) &= \frac{90}{2} \\ &\quad - \frac{1}{2} \sum_{n=1}^{90} \cos(2n^\circ). \end{aligned}$$

In the cosine sum, the terms for n and $90 - n$ satisfy $\cos(2n^\circ) + \cos(180^\circ - 2n^\circ) = 0$, and $\cos 90^\circ = 0$, so everything cancels except $\cos 180^\circ = -1$.

Hence the sum is $45 - \frac{1}{2}(-1) = 45.5$, and the mean is $\frac{45.5}{90} = \frac{91}{180}$.

Thus, the correct answer is **E**.

12. Suppose z is a complex number with positive imaginary part, with real part greater than 1, and with $|z| = 2$. In the complex plane, the four points $0, z, z^2$, and z^3 are the vertices of a quadrilateral with area 15. What is the imaginary part of z ?

A $\frac{3}{4}$

B 1

C $\frac{4}{3}$

D $\frac{3}{2}$

E $\frac{5}{3}$

Solution:

For vertices $0, z, z^2, z^3$ the shoelace formula gives area

$$\begin{aligned} & \frac{1}{2} |\operatorname{Im}(\bar{z}z^2 + \overline{z^2}z^3)| \\ &= \frac{1}{2} |\operatorname{Im}((|z|^2 + |z|^4)z)| \\ &= \frac{1}{2} (|z|^2 + |z|^4) \operatorname{Im}(z). \end{aligned}$$

With $|z| = 2$, this is $\frac{1}{2}(4 + 16) \operatorname{Im}(z) = 10 \operatorname{Im}(z)$. Setting $10 \operatorname{Im}(z) = 15$ gives $\operatorname{Im}(z) = \frac{3}{2}$. (Then $\operatorname{Re}(z) = \sqrt{4 - \frac{9}{4}} = \frac{\sqrt{7}}{2} > 1$, as required.)

Thus, the correct answer is **D**.

13. There are real numbers x, y, h , and k that satisfy the system of equations

$$x^2 + y^2 - 6x - 8y = h$$

$$x^2 + y^2 - 10x + 4y = k.$$

What is the minimum possible value of $h + k$?

A -54

B -46

C -34

D -16

E 16

Solution:

Adding the equations,

$$\begin{aligned} h + k &= 2x^2 + 2y^2 - 16x - 4y \\ &= 2(x - 4)^2 \\ &\quad + 2(y - 1)^2 - 34. \end{aligned}$$

Both squared terms are nonnegative, so the minimum occurs at $x = 4, y = 1$, giving $h + k = -34$.

Thus, the correct answer is **C**.

14. How many different remainders can result when the 100th power of an integer is divided by 125?

- ☐ A 1
- ☒ B 2
- ☐ C 5
- ☐ D 25
- ☐ E 125

Solution:

If n is coprime to 5, then since $\varphi(125) = 100$, Euler's theorem gives $n^{100} \equiv 1 \pmod{125}$. If n is a multiple of 5, then n^{100} is divisible by 5^{100} , hence by 125, leaving remainder 0.

So the only possible remainders are 0 and 1, which is 2 distinct values.

Thus, the correct answer is **B**.

15. A triangle in the coordinate plane has vertices $A(\log_2 1, \log_2 2)$, $B(\log_2 3, \log_2 4)$, and $C(\log_2 7, \log_2 8)$. What is the area of $\triangle ABC$?

A $\log_2 \frac{\sqrt{3}}{7}$

B $\log_2 \frac{3}{\sqrt{7}}$

C $\log_2 \frac{7}{\sqrt{3}}$

D $\log_2 \frac{11}{\sqrt{7}}$

E $\log_2 \frac{11}{\sqrt{3}}$

Solution:

The vertices are $A = (0, 1)$, $B = (\log_2 3, 2)$, $C = (\log_2 7, 3)$. By the shoelace formula,

$$\begin{aligned} & [\triangle ABC] \\ &= \frac{1}{2} |2 \log_2 3 - \log_2 7|. \end{aligned}$$

This equals $\frac{1}{2} \log_2 \frac{9}{7} = \log_2 \sqrt{\frac{9}{7}} = \log_2 \frac{3}{\sqrt{7}}$.

Thus, the correct answer is **B**.

16. A group of 16 people will be partitioned into indistinguishable 4-person committees. Each committee will have one chairperson and one secretary. The number of different ways to make these assignments can be written as $3^r M$, where r and M are positive integers and M is not divisible by 3. What is r ?

A 5

B 6

C 7

D 8

E 9

Solution:

The number of ways to split 16 people into 4 indistinguishable groups of 4 is $\frac{16!}{(4!)^4 4!}$.

Each committee then chooses a chairperson and a secretary in $4 \cdot 3 = 12$ ways, contributing 12^4 . So the total is $\frac{16!}{(4!)^4 4!} \cdot 12^4$.

Counting factors of 3 : $16!$ contributes $\lfloor \frac{16}{3} \rfloor + \lfloor \frac{16}{9} \rfloor = 6$. The denominator $(4!)^4 4!$ contributes $4 + 1 = 5$. And $12^4 = (2^2 \cdot 3)^4$ contributes 4. Thus $r = 6 - 5 + 4 = 5$.

Thus, the correct answer is **A**.

17. Integers a and b are randomly chosen without replacement from the set of integers with absolute value not exceeding 10. What is the probability that the polynomial $x^3 + ax^2 + bx + 6$ has 3 distinct integer roots?

A $\frac{1}{240}$

B $\frac{1}{221}$

C $\frac{1}{105}$

D $\frac{1}{84}$

E $\frac{1}{63}$

Solution:

The set has 21 integers, so there are $21 \cdot 20 = 420$ ordered choices of (a, b) . If the polynomial has distinct integer roots p, q, r , then $pqr = -6$, $a = -(p + q + r)$, and $b = pq + qr + rp$.

The triples of distinct integers with product -6 are $\{1, 2, -3\}$, $\{1, -2, 3\}$, $\{-1, 2, 3\}$, $\{-1, -2, -3\}$, and $\{1, -1, 6\}$. These give $(a, b) = (0, -7)$, $(-2, -5)$, $(-4, 1)$, $(6, 11)$, and $(-6, -1)$. The fourth has $b = 11 > 10$, so it is invalid; the other four are valid and distinct.

The probability is $\frac{4}{420} = \frac{1}{105}$.

Thus, the correct answer is **C**.

18. The Fibonacci numbers are defined by $F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. What is

$$\frac{F_2}{F_1} + \frac{F_4}{F_2} + \frac{F_6}{F_3} + \cdots + \frac{F_{20}}{F_{10}}?$$

A 318

B 319

C 320

D 321

E 322

Solution:

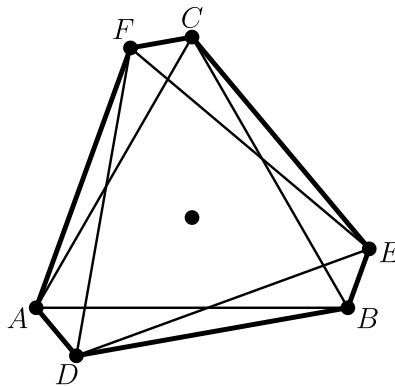
Since $F_{2k} = F_k L_k$ where L_k is the k th Lucas number, each term $\frac{F_{2k}}{F_k} = L_k$. The sum is

$$\begin{aligned} & L_1 + L_2 + \cdots + L_{10} \\ &= 1 + 3 + 4 + 7 + 11 + 18 \\ &\quad + 29 + 47 + 76 + 123 \\ &= 319. \end{aligned}$$

(Equivalently, $L_1 + \cdots + L_{10} = L_{12} - 3 = 322 - 3 = 319$.)

Thus, the correct answer is **B**.

19. Equilateral $\triangle ABC$ with side length 14 is rotated about its center by angle θ , where $0 < \theta < 60^\circ$, to form $\triangle DEF$. See the figure. The area of hexagon $ADBECF$ is $91\sqrt{3}$. What is $\tan \theta$?



- A $\frac{3}{4}$
- B $\frac{5\sqrt{3}}{11}$**
- C $\frac{4}{5}$
- D $\frac{11}{13}$
- E $\frac{7\sqrt{3}}{13}$

Solution:

The six vertices lie on the circumcircle of radius $R = \frac{14}{\sqrt{3}}$, so $R^2 = \frac{196}{3}$. Going around, the central angles alternate between θ (three times) and $120^\circ - \theta$ (three times). The cyclic-hexagon area is

$$\begin{aligned} & \frac{1}{2}R^2(3\sin\theta + 3\sin(120^\circ - \theta)) \\ &= 98(\sin\theta + \sin(120^\circ - \theta)). \end{aligned}$$

By sum-to-product, $\sin\theta + \sin(120^\circ - \theta) = 2\sin 60^\circ \cos(\theta - 60^\circ) = \sqrt{3}\cos(60^\circ - \theta)$. Setting the area to $91\sqrt{3}$ gives $\sqrt{3}\cos(60^\circ - \theta) = \frac{91\sqrt{3}}{98} =$

$$\frac{13\sqrt{3}}{14}, \text{ so } \cos(60^\circ - \theta) = \frac{13}{14} \text{ and } \sin(60^\circ - \theta) = \frac{3\sqrt{3}}{14}.$$

$$\text{Then } \tan(60^\circ - \theta) = \frac{3\sqrt{3}}{13}, \text{ and}$$

$$\tan \theta = \tan(60^\circ - (60^\circ - \theta))$$

$$= \frac{\sqrt{3} - \frac{3\sqrt{3}}{13}}{1 + \sqrt{3} \cdot \frac{3\sqrt{3}}{13}}$$

$$= \frac{\frac{10\sqrt{3}}{13}}{\frac{22}{13}}$$

$$= \frac{5\sqrt{3}}{11}.$$

Thus, the correct answer is **B**.

20. Suppose A, B , and C are points in the plane with $AB = 40$ and $AC = 42$, and let x be the length of the line segment from A to the midpoint of $\overline{BC}$. Define a function f by letting $f(x)$ be the area of $\triangle ABC$. Then the domain of f is an open interval (p, q) , and the maximum value r of $f(x)$ occurs at $x = s$. What is $p + q + r + s$?

A 909

B 910

C 911

D 912

E 913

Solution:

Let $a = BC$. The median length gives $x^2 = \frac{2 \cdot 1600 + 2 \cdot 1764 - a^2}{4} = \frac{6728 - a^2}{4}$.

The triangle inequality requires $2 < a < 82$, i.e. $4 < a^2 < 6724$, which translates to $1 < x < 41$. So $(p, q) = (1, 41)$.

With $AB = 40$ and $AC = 42$ fixed, the area $\frac{1}{2} \cdot 40 \cdot 42 \sin A$ is largest when $\angle A = 90^\circ$, giving $r = 840$. Then $a^2 = 40^2 + 42^2 = 3364$, so $x^2 = \frac{6728 - 3364}{4} = 841$, i.e. $s = 29$.

Thus $p + q + r + s = 1 + 41 + 840 + 29 = 911$.

Thus, the correct answer is **C**.

21. The measures of the smallest angles of three different right triangles sum to 90° . All three triangles have side lengths that are primitive Pythagorean triples. Two of them are 3-4-5 and 5-12-13. What is the perimeter of the third triangle?

A 40

B 126

C 154

D 176

E 208

Solution:

The smallest angles α, β of the 3-4-5 and 5-12-13 triangles have $\tan \alpha = \frac{3}{4}$ and $\tan \beta = \frac{5}{12}$. By the tangent addition formula,

$$\begin{aligned}\tan(\alpha + \beta) &= \frac{\frac{3}{4} + \frac{5}{12}}{1 - \frac{3}{4} \cdot \frac{5}{12}} \\ &= \frac{\frac{14}{12}}{\frac{33}{48}} \\ &= \frac{56}{33}.\end{aligned}$$

The third smallest angle γ satisfies $\gamma = 90^\circ - (\alpha + \beta)$, so $\tan \gamma = \frac{33}{56}$. The right triangle with legs 33 and 56 has hypotenuse $\sqrt{33^2 + 56^2} = \sqrt{4225} = 65$, a primitive triple. Its perimeter is $33 + 56 + 65 = 154$.

Thus, the correct answer is **C**.

22. Let $\triangle ABC$ be a triangle with integer side lengths and the property that $\angle B = 2\angle A$. What is the least possible perimeter of such a triangle?

A 13

B 14

C 15

D 16

E 17

Solution:

When $\angle B = 2\angle A$, the side lengths satisfy $b^2 = a(a + c)$, where $a = BC$, $b = CA$, $c = AB$. So $c = \frac{b^2 - a^2}{a}$ must be a positive integer, and the sides must form a valid triangle.

Also $b > a$, because $\angle B = 2\angle A > \angle A$. The perimeter is $a + b + c = b + \frac{b^2}{a} > 2b$. Therefore a perimeter below 15 would require $b \leq 6$. For $b = 2, 3, 4, 5, 6$, the divisors $a < b$ of b^2 give the possible pairs

$$\begin{aligned}(b, a) = & (2, 1), (3, 1), \\ & (4, 1), (4, 2), (5, 1), \\ & (6, 1), (6, 2), (6, 3), (6, 4).\end{aligned}$$

Substitution gives a degenerate or invalid triangle in every case except $(a, b) = (4, 6)$, which gives $c = 5$. Thus $(4, 5, 6)$ is the first valid triangle, and its perimeter is 15.

Thus, the correct answer is **C**.

23. A right pyramid has regular octagon $ABCDEFGH$ with side length 1 as its base and apex V . Segments $\overline{AV}$ and $\overline{DV}$ are perpendicular. What is the square of the height of the pyramid?

A 1

B $\frac{1 + \sqrt{2}}{2}$

C $\sqrt{2}$

D $\frac{3}{2}$

E $\frac{2 + \sqrt{2}}{3}$

Solution:

Let R be the circumradius of the octagon and L the length of each lateral edge, so $L^2 = h^2 + R^2$. Since $\angle AVD = 90^\circ$, $AD^2 = 2L^2$.

Vertices A and D are three steps apart, a central angle of 135° , so $AD^2 = 2R^2(1 - \cos 135^\circ) = R^2(2 + \sqrt{2})$. Setting $R^2(2 + \sqrt{2}) = 2(h^2 + R^2)$ gives $2h^2 = R^2\sqrt{2}$.

For a regular octagon of side 1, $R^2 = \frac{1}{2 \sin^2(22.5^\circ)} = \frac{2 + \sqrt{2}}{2}$. Therefore $h^2 = \frac{R^2\sqrt{2}}{2} = \frac{(2 + \sqrt{2})\sqrt{2}}{4} = \frac{2\sqrt{2} + 2}{4} = \frac{1 + \sqrt{2}}{2}$.

Thus, the correct answer is **B**.

24. What is the number of ordered triples (a, b, c) of positive integers, with $a \leq b \leq c \leq 9$, such that there exists a (non-degenerate) triangle $\triangle ABC$ with an integer inradius for which a, b , and c are the lengths of the altitudes from A to $\overline{BC}$, B to $\overline{AC}$, and C to $\overline{AB}$, respectively? (Recall that the inradius of a triangle is the radius of the largest possible circle that can be inscribed in the triangle.)

A 2

B 3

C 4

D 5

E 6

Solution:

Writing each side as $\frac{2[\triangle]}{h}$, the semiperimeter is $[\triangle]\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$, so the inradius $r = \frac{[\triangle]}{s}$ satisfies $\frac{1}{r} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$. We need this to be $\frac{1}{r}$ for a positive integer r , with the sides (proportional to $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$) forming a non-degenerate triangle, requiring $\frac{1}{a} < \frac{1}{b} + \frac{1}{c}$.

Because $a \leq b \leq c \leq 9$, the reciprocal sum is at least $\frac{3}{c} \geq \frac{1}{3}$, so the integer r is one of 1, 2, 3. Also $\frac{1}{a} < \frac{1}{r} \leq \frac{3}{a}$, so $r < a \leq 3r$. For each of these few values of r, a , substitute $b = a, a + 1, \dots, 9$ into

$$c = \frac{abr}{ab - ar - br}.$$

Keeping only integral c with $b \leq c \leq 9$ gives the complete list

r	(a, b, c)
1	$(2, 3, 6), (2, 4, 4), (3, 3, 3)$
2	$(4, 8, 8), (6, 6, 6)$
3	$(9, 9, 9).$

The triples $(2, 3, 6)$, $(2, 4, 4)$, and $(4, 8, 8)$ have $\frac{1}{a} = \frac{1}{b} + \frac{1}{c}$ and therefore give

degenerate triangles. The remaining triples are $(3, 3, 3)$, $(6, 6, 6)$, and $(9, 9, 9)$, so the answer is **3**.

Thus, the correct answer is **B**.

25. Pablo will decorate each of 6 identical white balls with either a striped or a dotted pattern, using either red or blue paint. He will decide on the color and pattern for each ball by flipping a fair coin for each of the 12 decisions he must make. After the paint dries, he will place the 6 balls in an urn. Frida will randomly select one ball from the urn and note its color and pattern. The events “the ball Frida selects is red” and “the ball Frida selects is striped” may or may not be independent, depending on the outcome of Pablo’s coin flips. The probability that these two events are independent can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. What is m ? (Recall that two events A and B are independent if $P(A \text{ and } B) = P(A) \cdot P(B)$.)

A 243

B 245

C 247

D 249

E 251

Solution:

Each ball is independently one of four equally likely types: red-striped, red-dotted, blue-striped, blue-dotted. Suppose among the 6 balls there are k red-striped, with R red and S striped in total. For Frida’s uniform pick, $P(\text{red}) = \frac{R}{6}$, $P(\text{striped}) = \frac{S}{6}$, and $P(\text{red and striped}) = \frac{k}{6}$. Independence means $\frac{k}{6} = \frac{R}{6} \cdot \frac{S}{6}$, i.e. $6k = RS$.

First count assignments with $R \in \{0, 6\}$ or $S \in \{0, 6\}$. Each of these four conditions leaves $2^6 = 64$ choices for the other attribute, and the four assignments at their pairwise intersections have been counted twice. Their union therefore contributes $4 \cdot 64 - 4 = 252$.

For $1 \leq R, S \leq 5$, the condition $6 \mid RS$ leaves only $(R, S) = (2, 3), (3, 2)$ or $(3, 4), (4, 3)$. In each case $k = \frac{RS}{6}$, and the four type counts are a permutation of $(1, 1, 2, 2)$. Thus each pair contributes $\frac{6!}{1!1!2!2!} = 180$ assignments. The favorable count is therefore $252 + 4 \cdot 180 = 972$. Out of $4^6 = 4096$ equally likely assignments, the probability is $\frac{972}{4096} = \frac{243}{1024}$, so $m = 243$.

Thus, the correct answer is **A**.

Problems: <https://live.poshenloh.com/past-contests/amc12/2024B>

