# AMC 12 SOLUTION ARCHIVE

# 2023

12A and 12B full explanations

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### About these explanations

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# 2023 AMC 12A Solutions

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**1.** Cities $A$ and $B$ are $45$ miles apart. Alicia lives in $A$ and Beth lives in $B$. Alicia bikes towards $B$ at $18$ miles per hour. Leaving at the same time, Beth bikes toward $A$ at $12$ miles per hour. How many miles from City $A$ will they be when they meet?

A $20$

B $24$

C $25$

D $26$

E $27$

## Solution:

The gap between them closes at $18 + 12 = 30$ miles per hour, so they meet after $\frac{45}{30} = 1.5$ hours.

In that time Alicia has ridden $18 \cdot 1.5 = 27$ miles from City $A$.

Thus, the correct answer is **E**.

**2.** The weight of $\frac{1}{3}$ of a large pizza together with $3\frac{1}{2}$ cups of orange slices is the same as the weight of $\frac{3}{4}$ of a large pizza together with $\frac{1}{2}$ cup of orange slices. A cup of orange slices weighs $\frac{1}{4}$ of a pound. What is the weight, in pounds, of a large pizza?

A $1\frac{4}{5}$

B $2$

C $2\frac{2}{5}$

D $3$

E $3\frac{3}{5}$

## Solution:

Let the pizza weigh $P$ pounds. Then $3\frac{1}{2}$ cups weigh $\frac{7}{2} \cdot \frac{1}{4} = \frac{7}{8}$ and $\frac{1}{2}$ cup weighs $\frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$.

The equation is

$$\frac{1}{3}P + \frac{7}{8} = \frac{3}{4}P + \frac{1}{8}.$$

Subtracting gives $\frac{7}{8} - \frac{1}{8} = \left(\frac{3}{4} - \frac{1}{3}\right)P$, so $\frac{3}{4} = \frac{5}{12}P$ and $P = \frac{9}{5} = 1\frac{4}{5}$.

Thus, the correct answer is **A**.

**3.** How many positive perfect squares less than $2023$ are divisible by $5$?

A $8$

B $9$

C $10$

D $11$

E $12$

## Solution:

A perfect square is divisible by $5$ only if its root is, so the squares are $(5k)^2 = 25k^2$.

Since $44^2 = 1936 < 2023 < 2025 = 45^2$, the root can be $5, 10, \ldots, 40$, which is $k = 1$ through $8$.

Thus, the correct answer is **A**.

**4.** How many digits are in the base-ten representation of $8^5 \cdot 5^{10} \cdot 15^5$?

A $14$

B $15$

C $16$

D $17$

E $18$

## Solution:

Writing everything in primes,

$$\begin{aligned} &8^5 \cdot 5^{10} \cdot 15^5 \\ &= 2^{15} \cdot 5^{10} \cdot 3^5 \cdot 5^5 \\ &= 3^5 \cdot 2^{15} \cdot 5^{15}. \end{aligned}$$

This equals $3^5 \cdot 10^{15} = 243 \cdot 10^{15}$, which is $243$ followed by $15$ zeros, for a total of $18$ digits.

Thus, the correct answer is **E**.

**5.** Janet rolls a standard $6$-sided die $4$ times and keeps a running total of the numbers she rolls. What is the probability that at some point, her running total will equal $3$?

A $\frac{2}{9}$

B $\frac{49}{216}$

C $\frac{25}{108}$

D $\frac{17}{72}$

E $\frac{13}{54}$

## Solution:

The running total is increasing, so it hits $3$ exactly when one of these disjoint openings occurs: a first roll of $3$; rolls $1, 2$; rolls $2, 1$; or rolls $1, 1, 1$.

Their probabilities are

$$\begin{aligned} &\frac{1}{6} + \frac{1}{36} + \frac{1}{36} + \frac{1}{216} \\ &= \frac{36 + 6 + 6 + 1}{216} \\ &= \frac{49}{216}. \end{aligned}$$

Thus, the correct answer is **B**.

**6.** Points $A$ and $B$ lie on the graph of $y = \log_2 x$. The midpoint of $\overline{AB}$ is $(6, 2)$. What is the positive difference between the $x$-coordinates of $A$ and $B$?

A $2\sqrt{11}$

B $4\sqrt{3}$

C $8$

D $4\sqrt{5}$

E $9$

## Solution:

Let the $x$-coordinates be $x_1$ and $x_2$. The midpoint gives $x_1 + x_2 = 12$, and the average of the $y$-values gives $\log_2 x_1 + \log_2 x_2 = 4$, so $x_1 x_2 = 2^4 = 16$.

Then

$$\begin{aligned} &|x_1 - x_2| \\ &= \sqrt{(x_1 + x_2)^2 - 4x_1 x_2} \\ &= \sqrt{144 - 64} \\ &= \sqrt{80} \\ &= 4\sqrt{5}. \end{aligned}$$

Thus, the correct answer is **D**.

**7.** A digital display shows the current date as an $8$-digit integer consisting of a $4$-digit year, followed by a $2$-digit month, followed by a $2$-digit date within the month. For example, Arbor Day this year is displayed as $20230428$. For how many dates in $2023$ will each digit appear an even number of times in the $8$-digit display for that date?

A $5$

B $6$

C $7$

D $8$

E $9$

## Solution:

The year contributes the digits $2, 0, 2, 3$, so $2$ appears twice while $0$ and $3$ each appear once. For every digit to end up with an even count, the four digits of the month and day must supply an odd number of $0$'s, an odd number of $3$'s, and an even number of every other digit.

With only four digits available, the month-day string must use exactly one $0$, one $3$, and a repeated pair of some digit. Checking valid months and days leaves nine dates: $01\text{-}13, 01\text{-}31, 02\text{-}23, 03\text{-}11, 03\text{-}22, 10\text{-}13, 10\text{-}31, 11\text{-}03$, and $11\text{-}30$.

Thus, the correct answer is **E**.

**8.** Maureen is keeping track of the mean of her quiz scores this semester. If Maureen scores an $11$ on the next quiz, her mean will increase by $1$. If she scores an $11$ on each of the next three quizzes, her mean will increase by $2$. What is the mean of her quiz scores currently?

A $4$

B $5$

C $6$

D $7$

E $8$

## Solution:

Let the current mean be $m$ over $n$ quizzes, so the total is $S = mn$. Adding one $11$ gives

$$\frac{mn + 11}{n + 1} = m + 1,$$

which simplifies to $m + n = 10$.

Adding three $11$'s gives

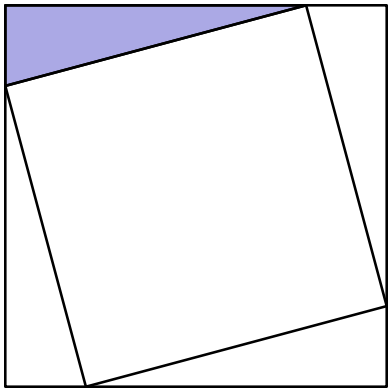
$$\frac{mn + 33}{n + 3} = m + 2,$$

which simplifies to $3m + 2n = 27$.

Solving $m + n = 10$ and $3m + 2n = 27$ gives $m = 7$.

Thus, the correct answer is **D**.

**9.** A square of area $2$ is inscribed in a square of area $3$, creating four congruent triangles, as shown below. What is the ratio of the shorter leg to the longer leg in the shaded right triangle?



A $\frac{1}{5}$

B $\frac{1}{4}$

C $2-\sqrt{3}$

D $\sqrt{3}-\sqrt{2}$

E $\sqrt{2}-1$

## Solution:

The outer square has side $\sqrt{3}$ and the inner square has side $\sqrt{2}$. Each triangle is right, with legs $p$ and $q$ along an outer side, so $p+q=\sqrt{3}$, and with hypotenuse an inner side, so $p^2+q^2=2$.

Then $(p+q)^2=3$ gives $2pq=1$, so $p$ and $q$ are the roots of $t^2-\sqrt{3}\,t+\frac{1}{2}=0$, namely $\dfrac{\sqrt{3}\pm 1}{2}$.

The ratio of shorter to longer leg is

$$\begin{aligned}&\frac{\sqrt{3}-1}{\sqrt{3}+1}\\ &=\frac{(\sqrt{3}-1)^2}{2}\\ &=2-\sqrt{3}.\end{aligned}$$

Thus, the correct answer is **C**.

**10.** Positive real numbers $x$ and $y$ satisfy $y^3 = x^2$ and $(y - x)^2 = 4y^2$. What is $x + y$?

A $12$

B $18$

C $24$

D $36$

E $42$

## Solution:

From $(y - x)^2 = 4y^2$ we get $y - x = \pm 2y$. The choice $y - x = 2y$ gives $x = -y < 0$, impossible, so $y - x = -2y$, meaning $x = 3y$.

Substituting into $y^3 = x^2 = 9y^2$ gives $y = 9$, hence $x = 27$ and $x + y = 36$.

Thus, the correct answer is **D**.

**11.** What is the degree measure of the acute angle formed by lines with slopes $2$ and $\frac{1}{3}$?

A $30$

B $37.5$

C $45$

D $52.5$

E $60$

## Solution:

The tangent of the angle between the lines is

$$\left|\frac{2-\frac{1}{3}}{1+2\cdot\frac{1}{3}}\right| = \left|\frac{\frac{5}{3}}{\frac{5}{3}}\right| = 1.$$

The acute angle with tangent $1$ is $45^\circ$.

Thus, the correct answer is **C**.

**12.** What is the value of

$$2^3 - 1^3 + 4^3 - 3^3 + 6^3 - 5^3 + \cdots + 18^3 - 17^3?$$

A $2023$

B $2679$

C $2941$

D $3159$

E $3235$

## Solution:

Group into pairs $(2k)^3 - (2k-1)^3$ for $k = 1, \ldots, 9$. Expanding,

$$(2k)^3 - (2k-1)^3 = 12k^2 - 6k + 1.$$

Summing for $k = 1$ to $9$, with $\sum k^2 = 285$ and $\sum k = 45$, gives

$$12 \cdot 285 - 6 \cdot 45 + 9 = 3420 - 270 + 9 = 3159.$$

Thus, the correct answer is **D**.

**13.** In a table tennis tournament every participant played every other participant exactly once. Although there were twice as many right-handed players as left-handed players, the number of games won by left-handed players was $40\%$ more than the number of games won by right-handed players. (There were no ties and no ambidextrous players.) What is the total number of games played?

A $15$

B $36$

C $45$

D $48$

E $66$

## Solution:

Let there be $L$ left-handed and $2L$ right-handed players, for $3L$ players and $\binom{3L}{2}$ games total.

If right-handers win $R$ games, left-handers win $1.4R$, so the total is $2.4R = \frac{12}{5}R$. For this to be an integer count, the total number of games must be a multiple of $12$.

The left-handers can win at most every game involving at least one left-hander, namely $\binom{L}{2} + 2L^2 = \frac{5L^2-L}{2}$. On the other hand, they must win $\frac{7}{12}\binom{3L}{2} = \frac{7L(3L-1)}{8}$ games. Comparing these quantities gives $21L - 7 \leq 20L - 4$, so $L \leq 3$.

Testing $L = 1, 2, 3$ gives totals $3, 15, 36$; only $36$ is a multiple of $12$. It is achievable: the $3$ left-handers can win all $18$ mixed games and their $3$ internal games, giving $21 = 1.4 \cdot 15$ wins.

Thus, the correct answer is **B**.

**14.** How many complex numbers satisfy the equation $z^5 = \overline{z}$, where $\overline{z}$ is the conjugate of the complex number $z$?

A $2$

B $3$

C $5$

D $6$

E $7$

## Solution:

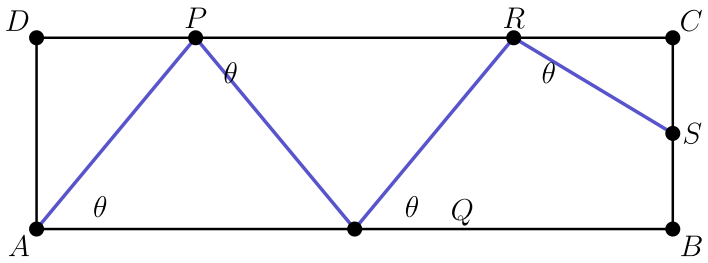
Taking magnitudes gives $|z|^5 = |z|$, so $|z| = 0$ or $|z| = 1$. The value $z = 0$ works, giving one solution.

If $|z| = 1$, multiply the equation by $z$ to get $z^6 = z\overline{z} = |z|^2 = 1$. This has $6$ distinct roots, all of modulus $1$.

Altogether there are $1 + 6 = 7$ solutions.

Thus, the correct answer is **E**.

**15.** Usain is walking for exercise by zigzagging across a $100$-meter by $30$-meter rectangular field, beginning at point $A$ and ending on the segment $\overline{BC}$. He wants to increase the distance walked by zigzagging as shown in the figure below ($APQRS$). What angle $\theta = \angle PAB = \angle QPC = \angle RQB = \cdots$ will produce a length that is $120$ meters? (Do not assume the zigzag path has exactly four segments as shown; there could be more or fewer.)




A $\arccos \frac{5}{6}$

B $\arccos \frac{4}{5}$

C $\arccos \frac{3}{10}$

D $\arcsin \frac{4}{5}$

E $\arcsin \frac{5}{6}$

## Solution:

Every segment of the zigzag makes angle $\theta$ with a horizontal side of the field. Therefore a segment of length $s$ advances $s\cos\theta$ meters horizontally. This remains true for the last segment even if it ends before crossing the full width of the field.

Adding the horizontal projections over the entire $120$-meter path gives $120\cos\theta = 100$.

Therefore $\cos\theta = \dfrac{5}{6}$, so $\theta = \arccos\dfrac{5}{6}$.

Thus, the correct answer is **A**.

**16.** Consider the set of complex numbers $z$ satisfying $|1+z+z^2|=4$. The maximum value of the imaginary part of $z$ can be written in the form $\frac{\sqrt{m}}{n}$, where $m$ and $n$ are relatively prime positive integers. What is $m+n$?

A $20$

B $21$

C $22$

D $23$

E $24$

## Solution:

Write $z=x+yi$. Then $1+z+z^2=(1+x+x^2-y^2)+y(1+2x)i$, and the constraint is

$$(1+x+x^2-y^2)^2 + y^2(1+2x)^2 = 16.$$

At a point where $y$ is maximal on this closed, bounded curve, implicit differentiation (or Lagrange multipliers) gives $(1+2x)\left(P+2y^2\right)=0$, where $P=1+x+x^2-y^2$. But $P+2y^2=x^2+x+1+y^2>0$, so $x=-\frac{1}{2}$.

Then $1+2x=0$, so the constraint reduces to $\left(\frac{3}{4}-y^2\right)^2=16$. Taking $\frac{3}{4}-y^2=-4$ gives $y^2=\frac{19}{4}$, so the maximum is $y=\dfrac{\sqrt{19}}{2}$.

Here $m=19$ and $n=2$, so $m+n=21$.

Thus, the correct answer is **B**.

**17.** Flora the frog starts at $0$ on the number line and makes a sequence of jumps to the right. In any one jump, independent of previous jumps, Flora leaps a positive integer distance $m$ with probability $\frac{1}{2^m}$. What is the probability that Flora will eventually land at $10$?

A $\frac{5}{512}$

B $\frac{45}{1024}$

C $\frac{127}{1024}$

D $\frac{511}{1024}$

E $\frac{1}{2}$

## Solution:

Let $a_n$ be the probability that Flora ever lands exactly on $n$, with $a_0 = 1$. Conditioning on the first jump,

$$a_n = \sum_{k=1}^{n} \frac{1}{2^k}\, a_{n-k}.$$

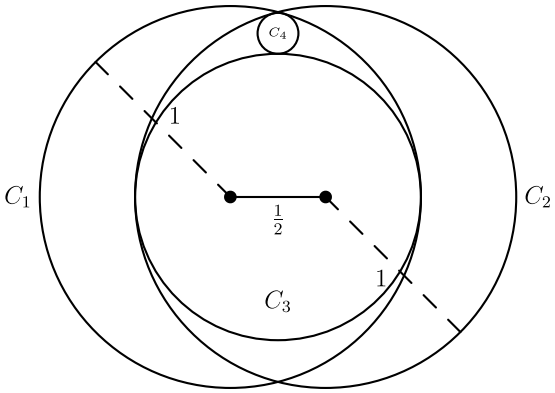
We prove by induction that $a_n = \frac{1}{2}$ for every $n \geq 1$. The case $n = 1$ is immediate. If the claim holds through $n-1$, then

$$\begin{aligned} a_n &= \frac{1}{2^n} a_0 + \sum_{k=1}^{n-1} \frac{1}{2^k} \cdot \frac{1}{2} \\ &= \frac{1}{2^n} + \frac{1}{2}\left(1 - \frac{1}{2^{n-1}}\right) \\ &= \frac{1}{2}. \end{aligned}$$

Hence the probability of landing on $10$ is $\frac{1}{2}$.

Thus, the correct answer is **E**.

**18.** Circle $C_1$ and $C_2$ each have radius $1$, and the distance between their centers is $\frac{1}{2}$. Circle $C_3$ is the largest circle internally tangent to both $C_1$ and $C_2$. Circle $C_4$ is internally tangent to both $C_1$ and $C_2$ and externally tangent to $C_3$. What is the radius of $C_4$?




A $\frac{1}{14}$

B $\frac{1}{12}$

C $\frac{1}{10}$

D $\frac{3}{28}$

E $\frac{1}{9}$

## Solution:

Put the centers at $O_1 = \left(-\frac{1}{4}, 0\right)$ and $O_2 = \left(\frac{1}{4}, 0\right)$. By symmetry $C_3$ is centered at the origin, and internal tangency to $C_1$ gives radius $1 - \frac{1}{4} = \frac{3}{4}$.

Let $C_4$ have radius $r$, centered at $(0, k)$ on the axis of symmetry. External tangency to $C_3$ gives $k = \frac{3}{4} + r$, and internal tangency to $C_1$ gives $\sqrt{\frac{1}{16} + k^2} = 1 - r$.

Substituting, $\frac{1}{16} + \left(\frac{3}{4} + r\right)^2 = (1 - r)^2$, which simplifies to $\frac{7}{2}r = \frac{3}{8}$, so $r = \frac{3}{28}$.

Thus, the correct answer is **D**.

**19.** What is the product of all the solutions to the equation

$$\log_{7x} 2023 \cdot \log_{289x} 2023 = \log_{2023x} 2023?$$

A $(\log_{2023} 7 \cdot \log_{2023} 289)^2$

B $\log_{2023} 7 \cdot \log_{2023} 289$

C $1$

D $\log_7 2023 \cdot \log_{289} 2023$

E $(\log_7 2023 \cdot \log_{289} 2023)^2$

## Solution:

Let $a = \log_{2023} 7$ and $b = \log_{2023} 289$. Since $2023 = 7 \cdot 289$, we have $a + b = 1$. Writing $t = \log_{2023} x$, each logarithm becomes a reciprocal, and the equation turns into

$$(1 + t) = (a + t)(b + t).$$

Expanding and using $a + b = 1$, the linear terms cancel, leaving $t^2 + (ab - 1) = 0$. Its two roots satisfy $t_1 + t_2 = 0$.

The corresponding solutions multiply to $x_1 x_2 = 2023^{t_1} \cdot 2023^{t_2} = 2023^{\,t_1 + t_2} = 2023^0 = 1$.

Thus, the correct answer is **C**.

**20.** Rows $1, 2, 3, 4,$ and $5$ of a triangular array of integers are shown below.

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 3 & & 1 & & \\ & 1 & & 5 & & 5 & & 1 & \\ 1 & & 7 & & 11 & & 7 & & 1 \end{array}$$

Each row after the first row is formed by placing a $1$ at each end of the row, and each interior entry is $1$ greater than the sum of the two numbers diagonally above it in the previous row. What is the units digit of the sum of the $2023$ numbers in the $2023$rd row?

A $1$

B $3$

C $5$

D $7$

E $9$

## Solution:

Let $S_n$ be the sum of row $n$. Each interior entry is $1$ more than the sum of the two entries above it, and summing over the row gives the recurrence

$$S_n = 2S_{n-1} + (n-2).$$

With $S_1 = 1$, this solves to $S_n = 2^n - n$ (check: $S_5 = 32 - 5 = 27 = 1 + 7 + 11 + 7 + 1$).

So $S_{2023} = 2^{2023} - 2023$. Since powers of $2$ cycle with units digits $2, 4, 8, 6$ and $2023 \equiv 3 \pmod 4$, $2^{2023}$ ends in $8$. Then $8 - 3 = 5$ gives units digit $5$.

Thus, the correct answer is **C**.

**21.** If $A$ and $B$ are vertices of a polyhedron, define the distance $d(A, B)$ to be the minimum number of edges of the polyhedron one must traverse in order to connect $A$ and $B$. For example, if $\overline{AB}$ is an edge of the polyhedron, then $d(A, B) = 1$, but if $\overline{AC}$ and $\overline{CB}$ are edges and $\overline{AB}$ is not an edge, then $d(A, B) = 2$. Let $Q$, $R$, and $S$ be randomly chosen distinct vertices of a regular icosahedron (regular polyhedron made up of $20$ equilateral triangles). What is the probability that $d(Q, R) > d(R, S)$?

A $\frac{7}{22}$

B $\frac{1}{3}$

C $\frac{3}{8}$

D $\frac{5}{12}$

E $\frac{1}{2}$

## Solution:

Fix $R$. Among the other $11$ vertices of the icosahedron, $5$ are at distance $1$, $5$ are at distance $2$, and $1$ (the antipode) is at distance $3$.

Choosing ordered distinct $Q, S$, the probability that $d(Q, R) = d(R, S)$ is

$$\frac{5 \cdot 4 + 5 \cdot 4}{11 \cdot 10} = \frac{40}{110} = \frac{4}{11}.$$

By the symmetry between $Q$ and $S$,

$$\begin{aligned} P(d(Q, R) > d(R, S)) &= \frac{1 - \frac{4}{11}}{2} \\ &= \frac{7}{22}. \end{aligned}$$

Thus, the correct answer is **A**.

**22.** Let $f$ be the unique function defined on the positive integers such that

$$\sum_{d|n} d \cdot f\left(\frac{n}{d}\right) = 1$$

for all positive integers $n$, where the sum is taken over all positive divisors of $n$. What is $f(2023)$?

A $-1536$

B $96$

C $108$

D $116$

E $144$

## Solution:

Setting $n = 1$ gives $f(1) = 1$. For a prime $p$, $n = p$ gives $f(p) + p \cdot f(1) = 1$, so $f(p) = 1 - p$. For $n = p^2$, $f(p^2) + p\, f(p) + p^2 f(1) = 1$ gives $f(p^2) = 1 - p$.

Since the defining relation is a Dirichlet convolution of multiplicative functions, $f$ is multiplicative. With $2023 = 7 \cdot 17^2$,

$$\begin{aligned} f(2023) &= f(7) \cdot f(17^2) \\ &= (1-7)(1-17) \\ &= (-6)(-16) \\ &= 96. \end{aligned}$$

Thus, the correct answer is **B**.

**23.** How many ordered pairs of positive real numbers $(a, b)$ satisfy the equation

$$(1+2a)(2+2b)(2a+b) = 32ab?$$

A $0$

B $1$

C $2$

D $3$

E an infinite number

## Solution:

By AM-GM, $1+2a \geq 2\sqrt{2a}$, $2+2b \geq 4\sqrt{b}$, and $2a+b \geq 2\sqrt{2ab}$. Multiplying,

$$\begin{aligned}(1+2a)(2+2b)(2a+b) &\geq 16\sqrt{2a}\cdot\sqrt{b}\cdot\sqrt{2ab} \\ &= 32ab.\end{aligned}$$

Equality requires $1 = 2a$, $2 = 2b$, and $2a = b$ simultaneously. These give $a = \frac{1}{2}, b = 1$, which are consistent, so there is exactly one solution.

Thus, the correct answer is **B**.

**24.** Let $K$ be the number of sequences $A_1, A_2, \ldots, A_n$ such that $n$ is a positive integer less than or equal to $10$, each $A_i$ is a subset of $\{1, 2, 3, \ldots, 10\}$, and $A_{i-1}$ is a subset of $A_i$ for each $i$ between $2$ and $n$, inclusive. For example, $\{\}, \{5, 7\}, \{2, 5, 7\}, \{2, 5, 7\}, \{2, 5, 6, 7, 9\}$ is one such sequence, with $n = 5$. What is the remainder when $K$ is divided by $10$?

A $1$

B $3$

C $5$

D $7$

E $9$

## Solution:

For a fixed length $n$, each element of $\{1, \ldots, 10\}$ independently either never appears or first appears in one of $A_1, \ldots, A_n$, giving $n + 1$ choices. Hence there are $(n + 1)^{10}$ chains of length $n$.

Summing,

$$K = \sum_{n=1}^{10} (n + 1)^{10} = \sum_{k=2}^{11} k^{10}.$$

Modulo $10$, the terms $k = 2, \ldots, 11$ reduce to $4, 9, 6, 5, 6, 9, 4, 1, 0, 1$, which sum to $45 \equiv 5$.

Thus, the correct answer is **C**.

**25.** There is a unique sequence of integers $a_1, a_2, \cdots a_{2023}$ such that

$$\tan 2023x = \frac{a_1 \tan x + a_3 \tan^3 x + a_5 \tan^5 x + \cdots + a_{2023} \tan^{2023} x}{1 + a_2 \tan^2 x + a_4 \tan^4 x \cdots + a_{2022} \tan^{2022} x}$$

whenever $\tan 2023x$ is defined. What is $a_{2023}$?

A $-2023$

B $-2022$

C $-1$

D $1$

E $2023$

## Solution:

By De Moivre, $(\cos x + i \sin x)^{2023} = \cos 2023x + i \sin 2023x$. Expanding the left side and taking the ratio of imaginary to real parts gives $\tan 2023x$ as the stated rational function of $\tan x$ after dividing numerator and denominator by $\cos^{2023} x$.

The coefficient $a_{2023}$ is the coefficient of $\tan^{2023} x$ in the numerator, which comes from the $k = 2023$ term:

$$\begin{aligned} a_{2023} &= (-1)^{\frac{2023-1}{2}} \binom{2023}{2023} \\ &= (-1)^{1011} \\ &= -1. \end{aligned}$$

Thus, the correct answer is **C**.

Problems: https://live.poshenloh.com/past-contests/amc12/2023A

# 2023 AMC 12B Solutions

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**1.** Mrs. Jones is pouring orange juice into four identical glasses for her four sons. She fills the first three glasses completely but runs out of juice when the fourth glass is only $\frac{1}{3}$ full. What fraction of a glass must Mrs. Jones pour from each of the first three glasses into the fourth glass so that all four glasses will have the same amount of juice?

A $\frac{1}{12}$

B $\frac{1}{4}$

C $\frac{1}{6}$

D $\frac{1}{8}$

E $\frac{2}{9}$

## Solution:

The total juice is $3 + \frac{1}{3} = \frac{10}{3}$ glasses. Split evenly, each glass should hold $\frac{10}{3} \div 4 = \frac{5}{6}$ of a glass. Each of the first three glasses must therefore give up $1 - \frac{5}{6} = \frac{1}{6}$.

Thus, the correct answer is **C**.

**2.** Carlos went to a sports store to buy running shoes. Running shoes were on sale, with prices reduced by $20\%$ on every pair of shoes. Carlos also knew that he had to pay a $7.5\%$ sales tax on the discounted price. He had $43$ dollars. What is the original (before discount) price of the most expensive shoes he could afford to buy?

A $46$

B $50$

C $48$

D $47$

E $49$

## Solution:

The final cost of a pair with original price $p$ is $0.8 \times 1.075 \times p = 0.86p$. Setting $0.86p \leq 43$ gives $p \leq 50$, so the most expensive affordable pair originally cost $50$ dollars.

Thus, the correct answer is **B**.

**3.** A $3$-$4$-$5$ right triangle is inscribed in circle $A$, and a $5$-$12$-$13$ right triangle is inscribed in circle $B$. What is the ratio of the area of circle $A$ to the area of circle $B$?

A $\frac{9}{25}$

B $\frac{1}{9}$

C $\frac{1}{5}$

D $\frac{25}{169}$

E $\frac{4}{25}$

## Solution:

The hypotenuse of an inscribed right triangle is a diameter, so circle $A$ has diameter $5$ and circle $B$ has diameter $13$. The ratio of areas is $\left(\frac{5}{13}\right)^2 = \frac{25}{169}$.

Thus, the correct answer is **D**.

**4.** Jackson's paintbrush makes a narrow strip with a width of $6.5$ millimeters. Jackson has enough paint to make a strip $25$ meters long. How many square centimeters of paper could Jackson cover with paint?

A $162{,}500$

B $162.5$

C $1{,}625$

D $1{,}625{,}000$

E $16{,}250$

## Solution:

Converting units, the strip is $0.65$ cm wide and $2500$ cm long, so its area is $0.65 \times 2500 = 1625$ square centimeters.

Thus, the correct answer is **C**.

**5.** You are playing a game. A $2 \times 1$ rectangle covers two adjacent squares (oriented either horizontally or vertically) of a $3 \times 3$ grid of squares, but you are not told which two squares are covered. Your goal is to find at least one square that is covered by the rectangle. A "turn" consists of you guessing a square, after which you are told whether that square is covered by the hidden rectangle. What is the minimum number of turns you need to ensure that at least one of your guessed squares is covered by the rectangle?

A $3$

B $5$

C $4$

D $8$

E $6$

## Solution:

A set of guessed squares is guaranteed to hit the domino if and only if the un-guessed squares contain no two adjacent squares, since otherwise the domino could hide on that adjacent pair. The largest set of pairwise non-adjacent squares in the $3 \times 3$ grid is the $5$-square checkerboard (four corners plus the center). So at most $5$ squares can be left unguessed, and you must guess $9 - 5 = 4$.

Thus, the correct answer is **C**.

**6.** When the roots of the polynomial

$$P(x) = (x-1)^1(x-2)^2 \cdot (x-3)^3 \cdots \cdot (x-10)^{10}$$

are removed from the number line, what remains is the union of $11$ disjoint open intervals. On how many of these intervals is $P(x)$ positive?

A $3$

B $7$

C $6$

D $4$

E $5$

## Solution:

The exponent of the factor $(x-k)$ is $k$, so the sign of $P$ changes at $x=k$ only when $k$ is odd, i.e. at $1, 3, 5, 7, 9$. For $x > 10$ every factor is positive, so $P > 0$. Sweeping left and flipping at each odd root, the positive intervals are $(10, \infty)$, $(9, 10)$, $(6, 7)$, $(5, 6)$, $(2, 3)$, and $(1, 2)$ — six intervals in all.

Thus, the correct answer is **C**.

**7.** For how many integers $n$ does the expression

$$\sqrt{\frac{\log(n^2) - (\log n)^2}{\log n - 3}}$$

represent a real number, where $\log$ denotes the base $10$ logarithm?

A $900$

B $3$

C $902$

D $2$

E $901$

## Solution:

Write $L = \log n$. Then $\log(n^2) - (\log n)^2 = 2L - L^2 = L(2 - L)$, and the fraction is $\frac{L(2-L)}{L-3}$. A sign chart shows this is $\geq 0$ exactly when $L \leq 0$ or $2 \leq L < 3$. Since $n$ is a positive integer, $L \leq 0$ forces $n = 1$, while $2 \leq L < 3$ gives $100 \leq n \leq 999$, which is $900$ values. In total $1 + 900 = 901$.

Thus, the correct answer is **E**.

**8.** How many nonempty subsets $B$ of $\{0, 1, 2, 3, \ldots, 12\}$ have the property that the number of elements in $B$ is equal to the least element of $B$? For example, $B = \{4, 6, 8, 11\}$ satisfies the condition.

A $256$

B $136$

C $108$

D $144$

E $156$

## Solution:

If the least element is $m$ $(m \geq 1)$, then $|B| = m$ and the remaining $m - 1$ elements come from $\{m + 1, \ldots, 12\}$, a set of size $12 - m$. The count is

$$\sum_{m \geq 1} \binom{12 - m}{m - 1} = \binom{11}{0} + \binom{10}{1} + \binom{9}{2} + \binom{8}{3} + \binom{7}{4} + \binom{6}{5},$$

which equals $1 + 10 + 36 + 56 + 35 + 6 = 144$.

Thus, the correct answer is **D**.

**9.** What is the area of the region in the coordinate plane defined by

$$||x| - 1| + ||y| - 1| \leq 1?$$

A $2$

B $8$

C $4$

D $15$

E $12$

## Solution:

Replacing $x, y$ by $|x|, |y|$, the condition $|x - 1| + |y - 1| \leq 1$ describes a diamond centered at $(1, 1)$ with diagonals of length $2$, hence area $2$. It lies entirely in the first quadrant (touching the axes only at single points), so reflecting across the two axes produces $4$ disjoint copies. The total area is $4 \times 2 = 8$.

Thus, the correct answer is **B**.

**10.** In the $xy$-plane, a circle of radius $4$ with center on the positive $x$-axis is tangent to the $y$-axis at the origin, and a circle with radius $10$ with center on the positive $y$-axis is tangent to the $x$-axis at the origin. What is the slope of the line passing through the two points at which these circles intersect?

A $\frac{2}{7}$

B $\frac{3}{7}$

C $\frac{2}{\sqrt{29}}$

D $\frac{1}{\sqrt{29}}$

E $\frac{2}{5}$

## Solution:

The circles are $(x-4)^2+y^2=16$ and $x^2+(y-10)^2=100$, i.e. $x^2+y^2=8x$ and $x^2+y^2=20y$. Subtracting gives $8x=20y$, so the intersection points lie on $y=\frac{2}{5}x$, which has slope $\frac{2}{5}$.

Thus, the correct answer is **E**.

**11.** What is the maximum area of an isosceles trapezoid that has legs of length $1$ and one base twice as long as the other?

A $\frac{5}{4}$

B $\frac{8}{7}$

C $\frac{5\sqrt{2}}{4}$

D $\frac{3}{2}$

E $\frac{3\sqrt{3}}{4}$

## Solution:

Let the bases be $a$ and $2a$. Each leg has horizontal offset $\frac{a}{2}$, so the height is $\sqrt{1-\frac{a^2}{4}}$ and the area is $A=\frac{3a}{2}\sqrt{1-\frac{a^2}{4}}$. Then $A^2=\frac{9}{4}\left(a^2-\frac{a^4}{4}\right)$, maximized when $a^2=2$. There the height is $\frac{1}{\sqrt{2}}$ and $A=\frac{3\sqrt{2}}{2}\cdot\frac{1}{\sqrt{2}}=\frac{3}{2}$.

Thus, the correct answer is **D**.

**12.** For complex numbers $u = a + bi$ and $v = c + di$, define the binary operation $\otimes$ by

$$u \otimes v = ac + bdi.$$

Suppose $z$ is a complex number such that $z \otimes z = z^2 + 40$. What is $|z|$?

A $2$

B $5$

C $\sqrt{5}$

D $\sqrt{10}$

E $5\sqrt{2}$

## Solution:

With $z = a + bi$, we have $z \otimes z = a^2 + b^2 i$ and $z^2 + 40 = (a^2 - b^2 + 40) + 2abi$. The real parts give $a^2 = a^2 - b^2 + 40$, so $b^2 = 40$. The imaginary parts give $b^2 = 2ab$, so $b = 2a$ and $a^2 = \frac{b^2}{4} = 10$. Then $|z|^2 = a^2 + b^2 = 50$, so $|z| = 5\sqrt{2}$.

Thus, the correct answer is **E**.

**13.** A rectangular box $P$ has distinct edge lengths $a$, $b$, and $c$. The sum of the lengths of all $12$ edges of $P$ is $13$, the sum of the areas of all $6$ faces of $P$ is $\frac{11}{2}$, and the volume of $P$ is $\frac{1}{2}$. What is the length of the longest interior diagonal connecting two vertices of $P$?

A $2$

B $\frac{3}{8}$

C $\frac{9}{8}$

D $\frac{9}{4}$

E $\frac{3}{2}$

## Solution:

From the edges, $4(a+b+c)=13$, so $a+b+c=\frac{13}{4}$. From the faces, $2(ab+bc+ca)=\frac{11}{2}$, so $ab+bc+ca=\frac{11}{4}$. Then

$$\begin{aligned} a^2+b^2+c^2 &= \left(\tfrac{13}{4}\right)^2 - 2\cdot\tfrac{11}{4} \\ &= \tfrac{169}{16} - \tfrac{88}{16} \\ &= \tfrac{81}{16}, \end{aligned}$$

so the diagonal is $\sqrt{\frac{81}{16}}=\frac{9}{4}$.

Thus, the correct answer is **D**.

**14.** For how many ordered pairs $(a, b)$ of integers does the polynomial $x^3 + ax^2 + bx + 6$ have $3$ distinct integer roots?

A $5$

B $6$

C $8$

D $7$

E $4$

## Solution:

By Vieta, the three distinct integer roots multiply to $-6$. The sets of three distinct integers with product $-6$ are $\{1, 2, -3\}, \{1, -2, 3\}, \{-1, 2, 3\}, \{-1, -2, -3\}$, and $\{1, -1, 6\}$. Each set determines $a = -(p + q + r)$ and $b = pq + qr + rp$, and all five give different pairs, so there are $5$ ordered pairs $(a, b)$.

Thus, the correct answer is **A**.

**15.** Suppose $a$, $b$, and $c$ are positive integers such that

$$\frac{a}{14} + \frac{b}{15} = \frac{c}{210}.$$

Which of the following statements are necessarily true?

I. If $\gcd(a, 14) = 1$ or $\gcd(b, 15) = 1$ or both, then $\gcd(c, 210) = 1$.

II. If $\gcd(c, 210) = 1$, then $\gcd(a, 14) = 1$ or $\gcd(b, 15) = 1$ or both.

III. $\gcd(c, 210) = 1$ if and only if $\gcd(a, 14) = \gcd(b, 15) = 1$.

A $\text{I}, \text{II}$, and $\text{III}$

B $\text{I}$ only

C $\text{I}$ and $\text{II}$ only

D $\text{III}$ only

E $\text{II}$ and $\text{III}$ only

## Solution:

Multiplying by $210$ gives $c = 15a + 14b$. Since $15 \equiv 1 \pmod{14}$, we get $c \equiv a \pmod{14}$, so $\gcd(c, 14) = 1$ iff $\gcd(a, 14) = 1$. Since $14 \equiv -1 \pmod{15}$, we get $c \equiv -b \pmod{15}$, so $\gcd(c, 15) = 1$ iff $\gcd(b, 15) = 1$. As $210 = 14 \cdot 15$ with $\gcd(14, 15) = 1$, statement III follows: $\gcd(c, 210) = 1$ iff both hold. Statement II is the forward implication of III, hence true. Statement I is false: if $\gcd(a, 14) = 1$ but $\gcd(b, 15) \neq 1$, then $\gcd(c, 15) \neq 1$, so $\gcd(c, 210) \neq 1$. Only II and III are true.

Thus, the correct answer is **E**.

**16.** In Coinland, there are three types of coins, each worth $6, 10,$ and $15.$ What is the sum of the digits of the maximum amount of money that is impossible to have?

A $8$

B $10$

C $7$

D $11$

E $9$

## Solution:

The amounts $30, 31, 32, 33, 34, 35$ are all attainable (for instance $30 = 6 \cdot 5, 31 = 6 + 10 + 15, 32 = 6 \cdot 2 + 10 \cdot 2, 33 = 6 \cdot 3 + 15, 34 = 6 \cdot 4 + 10, 35 = 10 + 10 + 15$). Adding $6$'s then reaches every larger amount. Checking below, $29$ is impossible: without a $15$-coin a sum of $6$'s and $10$'s is even, while with one $15$-coin the remaining $14$ cannot be written as $6a + 10b$; two $15$-coins already exceed $29.$ So the largest impossible amount is $29,$ whose digit sum is $2 + 9 = 11.$

Thus, the correct answer is **D**.

**17.** Triangle $ABC$ has side lengths in arithmetic progression, and the smallest side has length $6$. If the triangle has an angle of $120^\circ$, what is the area of $ABC$?

A $12\sqrt{3}$

B $8\sqrt{6}$

C $14\sqrt{2}$

D $20\sqrt{2}$

E $15\sqrt{3}$

## Solution:

Let the sides be $6,\ 6+d,\ 6+2d$. The $120^\circ$ angle faces the longest side, so $(6+2d)^2 = 6^2 + (6+d)^2 - 2\cdot 6\cdot(6+d)\cos 120^\circ$. Using $\cos 120^\circ = -\frac{1}{2}$ gives $3d^2 + 6d - 72 = 0$, so $d = 4$ and the sides are $6, 10, 14$. The area is $\frac{1}{2}\cdot 6\cdot 10\cdot \sin 120^\circ = 30\cdot\frac{\sqrt{3}}{2} = 15\sqrt{3}$.

Thus, the correct answer is **E**.

**18.** Last academic year Yolanda and Zelda took different courses that did not necessarily administer the same number of quizzes during each of the two semesters. Yolanda's average on all the quizzes she took during the first semester was $3$ points higher than Zelda's average on all the quizzes she took during the first semester. Yolanda's average on all the quizzes she took during the second semester was $18$ points higher than her average for the first semester and was again $3$ points higher than Zelda's average on all the quizzes Zelda took during her second semester. Which one of the following statements cannot possibly be true?

A Yolanda's quiz average for the academic year was $22$ points higher than Zelda's.

B Zelda's quiz average for the academic year was higher than Yolanda's.

C Yolanda's quiz average for the academic year was $3$ points higher than Zelda's.

D Zelda's quiz average for the academic year equaled Yolanda's.

E If Zelda had scored $3$ points higher on each quiz she took, then she would have had the same average for the academic year as Yolanda.

## Solution:

Subtracting Zelda's first-semester average from all four semester averages does not affect the comparison. We may therefore write the semester averages as $0, 18$ for Zelda and $3, 21$ for Yolanda. Let $p$ and $q$ be the fractions of Yolanda's and Zelda's quizzes, respectively, that occurred in the second semester. Their yearly-average difference is

$$(3 + 18p) - 18q = 3 + 18(p - q).$$

Because $0 < p, q < 1$, this difference is less than $21$, so it cannot be $22$.

The other options really can occur. Taking $(p, q) = (\frac{1}{4}, \frac{3}{4})$ makes Zelda's average higher; taking $p = q$ makes Yolanda's average $3$ points higher (and also verifies the last option after adding $3$ to every Zelda score); and taking $(p, q) = (\frac{1}{3}, \frac{1}{2})$ makes the yearly averages equal. Each displayed fraction can be realized by positive integer quiz counts.

Thus, the correct answer is **A**.

**19.** Each of $2023$ balls is placed in one of $3$ bins. Which of the following is closest to the probability that each of the bins will contain an odd number of balls?

A $\frac{2}{3}$

B $\frac{3}{10}$

C $\frac{1}{2}$

D $\frac{1}{3}$

E $\frac{1}{4}$

## Solution:

Counting assignments where all three bins are odd with the parity filter gives

$$\frac{1}{8}\sum_{S\subseteq\{1,2,3\}}(-1)^{|S|}(3-2|S|)^n = \frac{3^n-3}{4}$$

for odd $n$. Dividing by the $3^n$ total assignments, the probability is $\frac{3^n-3}{4\cdot 3^n}$, which for $n=2023$ is extremely close to $\frac{1}{4}$.

Thus, the correct answer is **E**.

**20.** Cyrus the frog jumps $2$ units in a direction, then $2$ more in another direction. What is the probability that he lands less than $1$ unit away from his starting position?

A $\frac{1}{6}$

B $\frac{1}{5}$

C $\frac{\sqrt{3}}{8}$

D $\frac{\arctan \frac{1}{2}}{\pi}$

E $\frac{2 \arcsin \frac{1}{4}}{\pi}$

## Solution:

Take the first jump as $(2, 0)$ and the second as $(2\cos\theta, 2\sin\theta)$ with $\theta$ uniform on $[0, 2\pi)$. The landing distance satisfies $R^2 = (2 + 2\cos\theta)^2 + (2\sin\theta)^2 = 8 + 8\cos\theta$. We need $R < 1$, i.e. $\cos\theta < -\frac{7}{8}$. The measure of such angles is $2\arccos\frac{7}{8}$, so the probability is $\frac{2\arccos\frac{7}{8}}{2\pi} = \frac{\arccos\frac{7}{8}}{\pi}$. Using $\arccos(1 - 2x^2) = 2\arcsin x$ with $x = \frac{1}{4}$ gives $\arccos\frac{7}{8} = 2\arcsin\frac{1}{4}$, so the probability is $\frac{2\arcsin\frac{1}{4}}{\pi}$.

Thus, the correct answer is **E**.

**21.** A lampshade is made in the form of the lateral surface of the frustum of a right circular cone. The height of the frustum is $3\sqrt{3}$ inches, its top diameter is $6$ inches, and its bottom diameter is $12$ inches. A bug is at the bottom of the lampshade and there is a glob of honey on the top edge of the lampshade at the spot farthest from the bug. The bug wants to crawl to the honey, but it must stay on the surface of the lampshade. What is the length in inches of its shortest path to the honey?

A $6 + 3\pi$

B $6 + 6\pi$

C $6\sqrt{3}$

D $6\sqrt{5}$

E $6\sqrt{3} + \pi$

## Solution:

Extend the frustum to a full cone. Since the radii are $3$ and $6$ with slant band $6$, the apex is slant distance $6$ from the top rim and $12$ from the bottom rim. The bottom circumference $12\pi$ unrolls to a sector of radius $12$ and angle $\frac{12\pi}{12} = \pi$. Place the bug at $(12, 0)$ in this pattern; the honey, halfway around the base, is at radius $6$ and angle $\frac{\pi}{2}$. The straight chord between them passes within radius $6$ (off the surface), so the geodesic goes tangent to the circle of radius $6$ : the tangent has length $\sqrt{12^2 - 6^2} = 6\sqrt{3}$ and touches at angle $\frac{\pi}{3}$, after which the path follows the arc of angle $\frac{\pi}{6}$ on radius $6$, of length $6 \cdot \frac{\pi}{6} = \pi$. The shortest path is $6\sqrt{3} + \pi$.

Thus, the correct answer is **E**.

**22.** A real-valued function $f$ has the property that for all real numbers $a$ and $b$,

$$f(a+b)+f(a-b) = 2f(a)f(b).$$

Which one of the following cannot be the value of $f(1)$?

A $0$

B $1$

C $-1$

D $2$

E $-2$

## Solution:

Setting $a=b=0$ gives $2f(0)=2f(0)^2$, so $f(0)=0$ or $f(0)=1$. If $f(0)=0$, then setting $b=0$ forces $f\equiv 0$, giving $f(1)=0$. Otherwise $f(0)=1$, and setting $a=b$ gives $f(2a)=2f(a)^2-1\geq -1$ for every $a$. In particular, with $a=\frac{1}{2}$, $f(1)=2f\left(\frac{1}{2}\right)^2-1\geq -1$. So $f(1)\geq -1$, and indeed every value in $[-1,\infty)$ is attainable (e.g. $f(x)=\cos(kx)$ or $\cosh(kx)$). Hence $-2$ is impossible.

Thus, the correct answer is **E**.

**23.** When $n$ standard six-sided dice are rolled, the product of the numbers rolled can be any of $936$ possible values. What is $n$?

A $11$

B $6$

C $8$

D $10$

E $9$

## Solution:

Each die contributes an exponent vector in the primes $2, 3, 5$ (face $1 \to (0,0,0)$, $2 \to (1,0,0), 3 \to (0,1,0), 4 \to (2,0,0), 5 \to (0,0,1), 6 \to (1,1,0)$), and a product is determined by the sum of these vectors.

Fix the exponent of $5$ by requiring exactly $k$ dice to show $5$, and put $m = n - k$. For an exponent $b$ of $3$, where $0 \le b \le m$, the possible exponents $a$ of $2$ are precisely $0, 1, \ldots, 2m - b$. If $a \le b$, use $a$ faces showing $6$ and $b - a$ showing $3$. If $a > b$, use $b$ faces showing $6$ and make the remaining $a - b$ factors of $2$ with faces $4$ and, if needed, one face $2$. The bound $a \le 2m - b$ says this uses at most $m$ dice; fill unused dice with $1$'s. Thus the number of exponent pairs is

$$\sum_{b=0}^{m} (2m - b + 1) = \frac{(m+1)(3m+2)}{2}.$$

Summing over $k = 0, 1, \ldots, n$, equivalently over $m = 0, 1, \ldots, n$, the number of distinct products is

$$\sum_{m=0}^{n} \frac{(m+1)(3m+2)}{2} = \frac{(n+1)^2(n+2)}{2}.$$

For $n = 11$ this is $\frac{12^2 \cdot 13}{2} = 936$, so $n = 11$.

Thus, the correct answer is **A**.

**24.** Suppose that $a, b, c$, and $d$ are positive integers satisfying all of the following relations.

$$\begin{aligned} abcd &= 2^6 \cdot 3^9 \cdot 5^7 \\ \operatorname{lcm}(a,b) &= 2^3 \cdot 3^2 \cdot 5^3 \\ \operatorname{lcm}(a,c) &= 2^3 \cdot 3^3 \cdot 5^3 \\ \operatorname{lcm}(a,d) &= 2^3 \cdot 3^3 \cdot 5^3 \\ \operatorname{lcm}(b,c) &= 2^1 \cdot 3^3 \cdot 5^2 \\ \operatorname{lcm}(b,d) &= 2^2 \cdot 3^3 \cdot 5^2 \\ \operatorname{lcm}(c,d) &= 2^2 \cdot 3^3 \cdot 5^2 \end{aligned}$$

What is $\gcd(a,b,c,d)$?

A $30$

B $45$

C $3$

D $15$

E $6$

## Solution:

Handle each prime separately using the exponents of $a, b, c, d$.

**Prime** $2$ (total $6$): $\max(b,c) = 1$ forces $a = 3$; then $b + c + d = 3$ with $\max(b,d) = \max(c,d) = 2$ gives $d = 2$ and $\{b,c\} = \{0,1\}$, so the minimum exponent is $0$.

**Prime** $3$ (total $9$): $\max(a,b) = 2$ with the other lcms equal to $3$ forces $c = d = 3$; then $a + b = 3$ with $\max(a,b) = 2$ gives $\{a,b\} = \{1,2\}$, so the minimum is $1$.

**Prime** $5$ (total $7$): $\max(a,b) = 3$ with $\max(b,c), \max(b,d), \max(c,d) = 2$ forces $a = 3$; then $b + c + d = 4$ with each $\leq 2$ and pairwise maxima $2$ gives two of them equal to $2$ and one equal to $0$, so the minimum is $0$.

Therefore $\gcd(a,b,c,d) = 2^0 \cdot 3^1 \cdot 5^0 = 3$.

Thus, the correct answer is **C**.

**25.** A regular pentagon with area $\sqrt{5}+1$ is printed on paper and cut out. The five vertices of the pentagon are folded into the center of the pentagon, creating a smaller pentagon. What is the area of the new pentagon?

A $4-\sqrt{5}$

B $\sqrt{5}-1$

C $8-3\sqrt{5}$

D $\dfrac{\sqrt{5}+1}{2}$

E $\dfrac{2+\sqrt{5}}{3}$

## Solution:

Let the original pentagon have circumradius $R$. Folding a vertex to the center creases along the perpendicular bisector of the segment from the center to that vertex, a line at distance $\frac{R}{2}$ from the center. The five creases bound a regular pentagon with apothem $\frac{R}{2}$, whereas the original has apothem $R\cos 36^\circ$. Areas scale as the square of the apothem, so the ratio is

$$\frac{\left(\frac{R}{2}\right)^2}{(R\cos 36^\circ)^2} = \frac{1}{4\cos^2 36^\circ}.$$

Since $\cos 36^\circ = \frac{1+\sqrt{5}}{4}$, this ratio is $\frac{4}{(1+\sqrt{5})^2} = \frac{4}{6+2\sqrt{5}} = \frac{2}{3+\sqrt{5}} = \frac{3-\sqrt{5}}{2}$. Multiplying by the original area $\sqrt{5}+1$ gives $\frac{(3-\sqrt{5})(\sqrt{5}+1)}{2} = \frac{2\sqrt{5}-2}{2} = \sqrt{5}-1.$

Thus, the correct answer is **B**.

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