

AMC 12 SOLUTION ARCHIVE

2022

12A and 12B full explanations

Contents

AMC 12A full solutions

27 pages

starts p. 2

AMC 12B full solutions

27 pages

starts p. 29

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[Open AMC 12A solution archive](#)

[Open AMC 12B solution archive](#)

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2022 AMC 12A Solutions

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1. What is the value of

$$3 + \frac{1}{3 + \frac{1}{3 + \frac{1}{3}}}$$

- A $\frac{31}{10}$
- B $\frac{49}{15}$
- C $\frac{33}{10}$
- D $\frac{109}{33}$**
- E $\frac{15}{4}$

Solution:

Simplify from the bottom. The innermost fraction is $3 + \frac{1}{3} = \frac{10}{3}$.

The next layer is $3 + \frac{1}{\frac{10}{3}} = 3 + \frac{3}{10} = \frac{33}{10}$.

Finally, $3 + \frac{1}{\frac{33}{10}} = 3 + \frac{10}{33} = \frac{109}{33}$.

Thus, the correct answer is **D**.

2. The sum of three numbers is 96. The first number is 6 times the third number, and the third number is 40 less than the second number. What is the absolute value of the difference between the first and second numbers?

- A 1
- B 2
- C 3
- D 4
- E 5

Solution:

Let the third number be t . Then the first is $6t$ and the second is $t + 40$. Their sum is

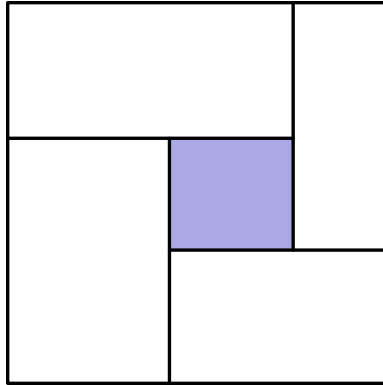
$$6t + (t + 40) + t = 8t + 40 = 96,$$

so $t = 7$.

The first number is 42 and the second is 47, so the difference has absolute value 5.

Thus, the correct answer is **E**.

3. Five rectangles, A , B , C , D , and E , are arranged in a square as shown below. These rectangles have dimensions 1×6 , 2×4 , 5×6 , 2×7 , and 2×3 , respectively. (The figure is not drawn to scale.) Which of the five rectangles is the shaded one in the middle?



- ☐ A A
- ☒ B B
- ☐ C C
- ☐ D D
- ☐ E E

Solution:

The five areas are 6, 8, 30, 14, and 6, which sum to 64. So the square is 8×8 .

Placing C (5×6) across the top left, D (2×7) up the right side, E (2×3) in the lower left, and A (1×6) along the bottom leaves a central 2×4 gap, which is exactly rectangle B .

Thus, the correct answer is **B**.

4. The least common multiple of a positive integer n and 18 is 180, and the greatest common divisor of n and 45 is 15. What is the sum of the digits of n ?

A 3

B 6

C 8

D 9

E 12

Solution:

Since $180 = 2^2 \cdot 3^2 \cdot 5$ and $18 = 2 \cdot 3^2$, the condition $\text{lcm}(n, 18) = 180$ forces n to contribute 2^2 and 5, with its power of 3 at most 2.

From $\text{gcd}(n, 45) = \text{gcd}(n, 3^2 \cdot 5) = 15 = 3 \cdot 5$, the power of 3 in n is exactly 1 and the power of 5 is at least 1.

Therefore $n = 2^2 \cdot 3 \cdot 5 = 60$, whose digits sum to 6.

Thus, the correct answer is **B**.

5. Let the *taxicab distance* between points (x_1, y_1) and (x_2, y_2) in the coordinate plane be given by $|x_1 - x_2| + |y_1 - y_2|$. For how many points P with integer coordinates is the taxicab distance between P and the origin less than or equal to 20?

A 441

B 761

C 841

D 921

E 924

Solution:

For each $k \geq 1$, the set $|x| + |y| = k$ contains exactly $4k$ lattice points, and $k = 0$ gives the single origin.

The total is

$$\begin{aligned} 1 + \sum_{k=1}^{20} 4k &= 1 + 4 \cdot \frac{20 \cdot 21}{2} \\ &= 1 + 840 = 841. \end{aligned}$$

Thus, the correct answer is **C**.

6. A data set consists of 6 (not distinct) positive integers: 1, 7, 5, 2, 5, and X . The average (arithmetic mean) of the 6 numbers equals a value in the data set. What is the sum of all positive values of X ?

A 10

B 26

C 32

D 36

E 40

Solution:

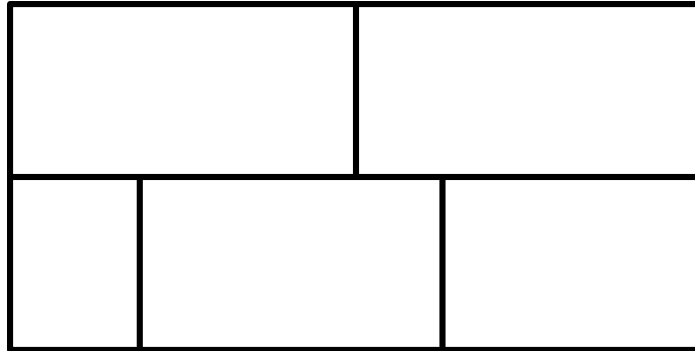
The known numbers sum to 20, so the mean is $\frac{20 + X}{6}$, which must equal an element of the set.

Setting it to 5 gives $X = 10$; to 7 gives $X = 22$; and to X itself gives $20 + X = 6X$, so $X = 4$. Values 1 and 2 give negative X .

The positive values are 10, 22, 4, summing to 36.

Thus, the correct answer is **D**.

7. A rectangle is partitioned into 5 regions as shown. Each region is to be painted a solid color - red, orange, yellow, blue, or green - so that regions that touch are painted different colors, and colors can be used more than once. How many different colorings are possible?



- A 120
- B 270
- C 360
- D 540**
- E 720

Solution:

The bottom-middle region shares a border with all four other regions. Color it first in 5 ways.

The top-left region borders it, giving 4 choices. Each of the three remaining regions borders exactly two already-colored regions, which have different colors, leaving 3 choices apiece.

The total is $5 \cdot 4 \cdot 3 \cdot 3 \cdot 3 = 540$.

Thus, the correct answer is **D**.

8. The infinite product

$$\sqrt[3]{10} \cdot \sqrt[3]{\sqrt[3]{10}} \cdot \sqrt[3]{\sqrt[3]{\sqrt[3]{10}}} \dots$$

evaluates to a real number. What is that number?

A $\sqrt{10}$

B $\sqrt[3]{100}$

C $\sqrt[4]{1000}$

D 10

E $10\sqrt[3]{10}$

Solution:

The k th factor is 10 raised to the k -fold cube root, namely $10^{\frac{1}{3^k}}$.

The product is 10 raised to

$$\begin{aligned} \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \dots &= \frac{\frac{1}{3}}{1 - \frac{1}{3}} \\ &= \frac{1}{2}. \end{aligned}$$

So the value is $10^{\frac{1}{2}} = \sqrt{10}$.

Thus, the correct answer is **A**.

9. On Halloween 31 children walked into the principal's office asking for candy. They can be classified into three types: some always lie; some always tell the truth; and some alternately lie and tell the truth. The alternaters arbitrarily choose their first response, either a lie or the truth, but each subsequent statement has the opposite truth value from its predecessor. The principal asked everyone the same three questions in this order.

"Are you a truth-teller?" The principal gave a piece of candy to each of the 22 children who answered yes.

"Are you an alternater?" The principal gave a piece of candy to each of the 15 children who answered yes.

"Are you a liar?" The principal gave a piece of candy to each of the 9 children who answered yes.

How many pieces of candy in all did the principal give to the children who always tell the truth?

☒ A 7

☐ B 12

☐ C 21

☐ D 27

☐ E 31

Solution:

To "Are you a truth-teller?" the truth-tellers and liars both answer yes, and only alternaters who lie on this question answer yes. To "Are you an alternater?" the liars answer yes, and among alternaters only those telling the truth on this question answer yes. To "Are you a liar?" only alternaters lying on this question answer yes.

Split the alternaters by first response. Those starting with a lie answer (lie, truth, lie), so they say yes to all three questions; those starting truthful answer (truth, lie, truth) and say yes to none of the three. The 9 yeses on the last question are exactly the lie-first alternaters, so there are 9 of them.

The second question's 15 yeses are the liars plus these 9, so there are 6 liars. The first question's 22 yeses are truth-tellers plus liars plus the 9, so the truth-tellers number $22 - 6 - 9 = 7$.

Truth-tellers answer yes only to the first question, receiving one candy each, for $7 \cdot 1 = 7$ pieces.

Thus, the correct answer is **A**.

10. What is the number of ways the numbers from 1 to 14 can be split into 7 pairs such that for each pair, the greater number is at least 2 times the smaller number?

A 108

B 120

C 126

D 132

E 144

Solution:

Any number 8 or larger cannot be a smaller element (its double exceeds 14), so 8–14 are all larger elements and 1–7 are all smaller elements.

Match each smaller s to a larger $g \geq 2s$. Processing from the most restrictive: $s = 7$ forces $g = 14$ (1 way); then $s = 6$ has $\{12, 13\}$ left (2); $s = 5$ has 3; $s = 4$ has 4; $s = 3$ has 3; $s = 2$ has 2; $s = 1$ has 1.

The number of matchings is $1 \cdot 2 \cdot 3 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 144$.

Thus, the correct answer is **E**.

11. What is the product of all real numbers x such that the distance on the number line between $\log_6 x$ and $\log_6 9$ is twice the distance on the number line between $\log_6 10$ and 1?

A 10

B 18

C 25

D 36

E 81

Solution:

The right-hand distance is $|\log_6 10 - 1| = \log_6 \frac{5}{3}$, so twice it is $2 \log_6 \frac{5}{3} = \log_6 \frac{25}{9}$.

Thus $\left| \log_6 \frac{x}{9} \right| = \log_6 \frac{25}{9}$, giving $\frac{x}{9} = \frac{25}{9}$ or $\frac{x}{9} = \frac{9}{25}$, so $x = 25$ or $x = \frac{81}{25}$.

Their product is $25 \cdot \frac{81}{25} = 81$.

Thus, the correct answer is **E**.

12. Let M be the midpoint of $\overline{AB}$ in regular tetrahedron $ABCD$. What is $\cos(\angle CMD)$?

A $\frac{1}{4}$

B $\frac{1}{3}$

C $\frac{2}{5}$

D $\frac{1}{2}$

E $\frac{\sqrt{3}}{2}$

Solution:

Take edge length 1. Since M is the midpoint of $\overline{AB}$, segments CM and DM are altitudes of the equilateral faces, each of length $\frac{\sqrt{3}}{2}$. Also $CD = 1$.

By the Law of Cosines in $\triangle CMD$,

$$\begin{aligned}\cos(\angle CMD) &= \frac{\frac{3}{4} + \frac{3}{4} - 1}{2 \cdot \frac{3}{4}} \\ &= \frac{\frac{1}{2}}{\frac{3}{2}} = \frac{1}{3}.\end{aligned}$$

Thus, the correct answer is **B**.

13. Let $\mathcal{R}$ be the region in the complex plane consisting of all complex numbers z that can be written as the sum of complex numbers z_1 and z_2 , where z_1 lies on the segment with endpoints 3 and $4i$, and z_2 has magnitude at most 1 . What integer is closest to the area of $\mathcal{R}$?

A 13

B 14

C 15

D 16

E 17

Solution:

Adding a disk of radius 1 to every point of the segment sweeps out all points within distance 1 of it. The segment from 3 to $4i$ has length $\sqrt{3^2 + 4^2} = 5$.

This “stadium” is a 5×2 rectangle plus two half-disks of radius 1 , with area

$$5 \cdot 2 + \pi(1)^2 = 10 + \pi \approx 13.14.$$

The closest integer is **13**.

Thus, the correct answer is **A**.

14. What is the value of

$$(\log 5)^3 + (\log 20)^3 \\ + (\log 8)(\log 0.25)$$

where $\log$ denotes the base-ten logarithm?

A $\frac{3}{2}$

B $\frac{7}{4}$

C 2

D $\frac{9}{4}$

E 3

Solution:

Let $u = \log 2$. Then $\log 5 = 1 - u$, $\log 20 = 1 + u$, $\log 8 = 3u$, and $\log 0.25 = -2u$.

With $a = 1 - u$, $b = 1 + u$, we have $a + b = 2$ and $ab = 1 - u^2$, so

$$\begin{aligned} & a^3 + b^3 \\ &= (a + b)((a + b)^2 - 3ab) \\ &= 2(4 - 3(1 - u^2)) \\ &= 2 + 6u^2. \end{aligned}$$

The last term is $(3u)(-2u) = -6u^2$, so the total is $2 + 6u^2 - 6u^2 = 2$.

Thus, the correct answer is **C**.

15. The roots of the polynomial $10x^3 - 39x^2 + 29x - 6$ are the height, length, and width of a rectangular box (right rectangular prism). A new rectangular box is formed by lengthening each edge of the original box by 2 units. What is the volume of the new box?

A $\frac{24}{5}$

B $\frac{42}{5}$

C $\frac{81}{5}$

D 30

E 48

Solution:

Let the roots be r, s, t . By Vieta's formulas, $r + s + t = \frac{39}{10}$, $rs + rt + st = \frac{29}{10}$, and $rst = \frac{6}{10} = \frac{3}{5}$.

The new volume is

$$\begin{aligned} & (r + 2)(s + 2)(t + 2) \\ &= rst + 2(rs + rt + st) \\ & \quad + 4(r + s + t) + 8 \\ &= \frac{3}{5} + \frac{58}{10} \\ & \quad + \frac{156}{10} + 8 \\ &= 30. \end{aligned}$$

Thus, the correct answer is **D**.

16. A *triangular number* is a positive integer that can be expressed in the form $t_n = 1 + 2 + 3 + \cdots + n$, for some positive integer n . The three smallest triangular numbers that are also perfect squares are $t_1 = 1 = 1^2$, $t_8 = 36 = 6^2$, and $t_{49} = 1225 = 35^2$. What is the sum of the digits of the fourth smallest triangular number that is also a perfect square?

A 6

B 9

C 12

D 18

E 27

Solution:

If $t_n = y^2$, then $\frac{n(n+1)}{2} = y^2$, or $(2n+1)^2 - 8y^2 = 1$. The positive Pell solutions occur successively by multiplying $(2n+1) + y\sqrt{8}$ by $3 + \sqrt{8}$.

Starting from $(2n+1, y) = (3, 1)$, this gives $(17, 6)$, $(99, 35)$, and then $(577, 204)$. Thus the fourth value has $n = 288$ and equals $204^2 = 41616$.

The sum of its digits is $4 + 1 + 6 + 1 + 6 = 18$.

Thus, the correct answer is **D**.

17. Suppose a is a real number such that the equation

$$a \cdot (\sin x + \sin(2x)) = \sin(3x)$$

has more than one solution in the interval $(0, \pi)$. The set of all such a can be written in the form $(p, q) \cup (q, r)$, where p, q , and r are real numbers with $p < q < r$. What is $p + q + r$?

A ☒ -4

B ☐ -1

C ☐ 0

D ☐ 1

E ☐ 4

Solution:

Since $\sin x \neq 0$ on $(0, \pi)$, divide by $\sin x$:

$$\begin{aligned} a(1 + 2 \cos x) &= 4 \cos^2 x - 1 \\ &= (2 \cos x - 1)(2 \cos x + 1). \end{aligned}$$

When $\cos x = -\frac{1}{2}$ (that is, $x = \frac{2\pi}{3}$) both sides vanish, so this is a solution for every a .

Otherwise we may cancel $1 + 2 \cos x$ to get $a = 2 \cos x - 1$, i.e. $\cos x = \frac{a+1}{2}$.

This yields a second solution in $(0, \pi)$ exactly when $-1 < \frac{a+1}{2} < 1$, that is $a \in (-3, 1)$, and it is distinct from $x = \frac{2\pi}{3}$ unless $a = -2$.

So more than one solution occurs for $a \in (-3, -2) \cup (-2, 1)$, giving $p + q + r = -3 - 2 + 1 = -4$.

Thus, the correct answer is **A**.

18. Let T_k be the transformation of the coordinate plane that first rotates the plane k degrees counterclockwise around the origin and then reflects the plane across the y -axis. What is the least positive integer n such that performing the sequence of transformations $T_1, T_2, T_3, \dots, T_n$ returns the point $(1, 0)$ back to itself?

A 359

B 360

C 719

D 720

E 721

Solution:

Rotating a point at angle θ by k° gives $\theta + k$, and reflecting across the y -axis sends angle ϕ to $180 - \phi$. So T_k sends θ to $(180 - k) - \theta$.

Starting from $(1, 0)$ at angle 0, applying $T_1, T_2, \dots$ gives angles $179, -1, 178, -2, 177, \dots$. After an even number $2m$ of steps the angle is $-m$, and after an odd number $2m + 1$ it is $179 - m$.

For the point to return, the angle must be a multiple of 360° . The even case needs $m = 360$, i.e. $n = 720$. The odd case needs $179 - m = 0$, i.e. $m = 179$ and $n = 359$, where the net reflection fixes $(1, 0)$.

The least such n is 359.

Thus, the correct answer is **A**.

19. Suppose that 13 cards numbered $1, 2, 3, \dots, 13$ are arranged in a row. The task is to pick them up in numerically increasing order, working repeatedly from left to right. In the example below, cards 1, 2, 3 are picked up on the first pass, 4 and 5 on the second pass, 6 on the third pass, 7, 8, 9, 10 on the fourth pass, and 11, 12, 13 on the fifth pass. For how many of the $13!$ possible orderings of the cards will the 13 cards be picked up in exactly two passes?

7	11	8	6	4	5	9	12	1	13	10	2	3
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- A 4082
- B 4095
- C 4096
- D 8178**
- E 8191

Solution:

Let $\text{pos}(k)$ be the position of card k . A fresh pass is needed exactly when $\text{pos}(k+1) < \text{pos}(k)$, so the number of passes is one more than the number of descents in the sequence $\text{pos}(1), \text{pos}(2), \dots, \text{pos}(13)$.

To build a permutation with at most one descent, choose the entries before the possible descent and write both chosen blocks in increasing order. There are 2^{13} subsets. The 14 initial segments $\emptyset, \{1\}, \dots, \{1, \dots, 13\}$ produce no descent; every other subset produces a unique permutation with one descent. Hence the count is $2^{13} - 14 = 8178$.

Thus, the correct answer is **D**.

20. Isosceles trapezoid $ABCD$ has parallel sides $\overline{AD}$ and $\overline{BC}$, with $BC < AD$ and $AB = CD$. There is a point P in the plane such that $PA = 1$, $PB = 2$, $PC = 3$, and $PD = 4$. What is $\frac{BC}{AD}$?

A $\frac{1}{4}$

B $\frac{1}{3}$

C $\frac{1}{2}$

D $\frac{2}{3}$

E $\frac{3}{4}$

Solution:

Place the trapezoid symmetric about the y -axis: $A = (-p, 0)$, $D = (p, 0)$, $B = (-q, h)$, $C = (q, h)$, with $P = (x, y)$.

Then $PA^2 - PD^2 = 4px = 1 - 16 = -15$ and $PB^2 - PC^2 = 4qx = 4 - 9 = -5$. Dividing gives $\frac{p}{q} = 3$.

Since $AD = 2p$ and $BC = 2q$, we get $\frac{BC}{AD} = \frac{q}{p} = \frac{1}{3}$.

Thus, the correct answer is **B**.

21. Let $P(x) = x^{2022} + x^{1011} + 1$. Which of the following polynomials divides $P(x)$?

A $x^2 - x + 1$

B $x^2 + x + 1$

C $x^4 + 1$

D $x^6 - x^3 + 1$

E $x^6 + x^3 + 1$

Solution:

If ζ is a root of a divisor, then $P(\zeta) = \zeta^{2022} + \zeta^{1011} + 1 = 0$ is required.

The roots of $x^6 + x^3 + 1$ are the primitive 9th roots of unity, so $\zeta^9 = 1$. Reducing exponents modulo 9, $2022 \equiv 6$ and $1011 \equiv 3$, giving $\zeta^6 + \zeta^3 + 1$. Here $\omega = \zeta^3$ is a primitive cube root of unity, so this equals $\omega^2 + \omega + 1 = 0$.

The other four options fail: substituting their roots yields nonzero values (for instance, the primitive cube roots of unity give $P = 3$).

Thus, the correct answer is **E**.

22. Let c be a real number, and let z_1, z_2 be the two complex numbers satisfying the quadratic $z^2 - cz + 10 = 0$. Points $z_1, z_2, \frac{1}{z_1}$, and $\frac{1}{z_2}$ are the vertices of a (convex) quadrilateral Q in the complex plane. When the area of Q obtains its maximum value, c is the closest to which of the following?

A 4.5

B 5

C 5.5

D 6

E 6.5

Solution:

If the roots are non-real, then $z_1 = \sqrt{10}e^{i\theta}$ and $z_2 = \overline{z_1}$, since $z_1 z_2 = 10$. Then $\frac{1}{z_1} = \frac{1}{\sqrt{10}}e^{-i\theta}$ and $\frac{1}{z_2} = \frac{1}{\sqrt{10}}e^{i\theta}$.

The two vertical sides of this trapezoid have lengths $2\sqrt{10}\sin\theta$ and $\frac{2\sin\theta}{\sqrt{10}}$, and their horizontal separation is $(\sqrt{10} - \frac{1}{\sqrt{10}})\cos\theta$. Hence its area is

$$\frac{99}{10}\sin\theta\cos\theta = \frac{99}{20}\sin 2\theta,$$

which is maximized at $\theta = 45^\circ$.

Then $c = z_1 + z_2 = 2\sqrt{10}\cos 45^\circ = 2\sqrt{5} \approx 4.47$, closest to 4.5.

Thus, the correct answer is **A**.

23. Let h_n and k_n be the unique relatively prime positive integers such that

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \frac{h_n}{k_n}.$$

Let L_n denote the least common multiple of the numbers $1, 2, 3, \dots, n$. For how many integers n with $1 \leq n \leq 22$ is $k_n < L_n$?

- ☐ A 0
- ☐ B 3
- ☐ C 7
- ☒ D 8
- ☐ E 10

Solution:

Always $k_n \mid L_n$, so $k_n < L_n$ exactly when some prime p divides both L_n and the numerator $N = \sum_{k=1}^n \frac{L_n}{k}$ (i.e. a prime cancels).

For a prime p with maximal power $p^a \leq n$, only the terms with $v_p(k) = a$ keep p out of $\frac{L_n}{k}$; all others are divisible by p . So p cancels iff $\sum_{v_p(k)=a} \frac{L_n}{k} \equiv 0 \pmod{p}$.

Applying this test recursively with $H_n = H_{n-1} + \frac{1}{n}$ gives $k_n = L_n$ for $1 \leq n \leq 5$, $k_n = \frac{L_n}{3}$ for $6 \leq n \leq 8$, and $k_n = L_n$ again for $9 \leq n \leq 17$. Finally, the ratios $\frac{L_n}{k_n}$ for $n = 18, 19, 20, 21, 22$ are 3, 3, 15, 45, 45, respectively. Thus cancellation occurs precisely for $n = 6, 7, 8, 18, 19, 20, 21, 22$, which is 8 values.

Thus, the correct answer is **D**.

24. How many strings of length 5 formed from the digits 0, 1, 2, 3, 4 are there such that for each $j \in \{1, 2, 3, 4\}$, at least j of the digits are less than j ? (For example, 02214 satisfies the condition because it contains at least 1 digit less than 1, at least 2 digits less than 2, at least 3 digits less than 3, and at least 4 digits less than 4. The string 23404 does not satisfy the condition because it does not contain at least 2 digits less than 2.)

A 500

B 625

C 1089

D 1199

E 1296

Solution:

Sort the five digits as $d_{(1)} \leq d_{(2)} \leq \dots \leq d_{(5)}$. The requirement “at least j digits less than j ” is equivalent to $d_{(j)} \leq j - 1$ for $j = 1, 2, 3, 4$, i.e. $d_{(1)} = 0$, $d_{(2)} \leq 1$, $d_{(3)} \leq 2$, $d_{(4)} \leq 3$ (with $d_{(5)} \leq 4$ automatic).

These strings are exactly the parking functions of length 5. To count them, arrange 6 parking spaces in a circle and let 5 labeled cars choose arbitrary preferred spaces.

Each car moves forward to the first open space. Among the 6^5 preference strings, rotating all preferences cycles the unique empty space through all 6 positions.

Therefore exactly $\frac{6^5}{6} = 6^4 = 1296$ strings leave a specified space empty. Choosing that space as the extra sixth space gives precisely the sorted inequalities above.

Thus, the correct answer is **E**.

25. A circle with integer radius r is centered at (r, r) . Distinct line segments of length c_i connect points $(0, a_i)$ to $(b_i, 0)$ for $1 \leq i \leq 14$ and are tangent to the circle, where a_i , b_i , and c_i are all positive integers and $c_1 \leq c_2 \leq \dots \leq c_{14}$. What is the ratio $\frac{c_{14}}{c_1}$ for the least possible value of r ?

- A $\frac{21}{5}$
- B $\frac{85}{13}$
- C 7
- D $\frac{39}{5}$
- E 17

Solution:

The circle centered (r, r) with radius r is tangent to both axes. A segment from $(0, a)$ to $(b, 0)$ with $a^2 + b^2 = c^2$ is tangent to it when r equals either the inradius $\frac{a+b-c}{2}$ or the semiperimeter $\frac{a+b+c}{2}$ of the right triangle with legs a, b .

In the inradius case, put $x = a - 2r$ and $y = b - 2r$. Then $xy = 2r^2$, and every positive divisor x determines one oriented segment, with $a = x + 2r$, $b = \frac{2r^2}{x} + 2r$, and $c = x + \frac{2r^2}{x} + 2r$. Thus there are exactly $d(2r^2)$ such segments.

For $r = 1, 2, 3, 4, 5$, these counts are 2, 4, 6, 6, 6, and no semiperimeter case is possible because the smallest integer right triangle has semiperimeter 6. At $r = 6$, $d(72) = 12$, and the 3-4-5 triangle contributes two more oriented segments. Hence 6 is the least possible radius and gives exactly 14 segments.

The two semiperimeter segments have $c = 5$. In the inradius family, $c = x + \frac{72}{x} + 12$ is largest at $x = 1$ or 72, giving $c = 85$. Therefore $c_1 = 5$, $c_{14} = 85$, and $\frac{c_{14}}{c_1} = 17$.

Thus, the correct answer is **E**.

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1. Define $x \diamond y$ to be $|x - y|$ for all real numbers x and y . What is the value of

$$(1 \diamond (2 \diamond 3)) - ((1 \diamond 2) \diamond 3)?$$

A -2

B -1

C 0

D 1

E 2

Solution:

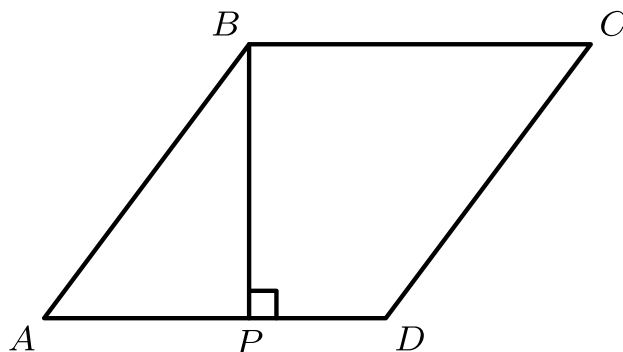
Since $2 \diamond 3 = |2 - 3| = 1$, we get $1 \diamond (2 \diamond 3) = 1 \diamond 1 = 0$.

Since $1 \diamond 2 = |1 - 2| = 1$, we get $(1 \diamond 2) \diamond 3 = 1 \diamond 3 = |1 - 3| = 2$.

The value is $0 - 2 = -2$.

Thus, the correct answer is **A**.

2. In rhombus $ABCD$, point P lies on segment $\overline{AD}$ so that $\overline{BP} \perp \overline{AD}$, $AP = 3$, and $PD = 2$. What is the area of $ABCD$? (Note: the figure is not drawn to scale.)



- ☐ A $3\sqrt{5}$
- ☐ B 10
- ☐ C $6\sqrt{5}$
- ☒ D 20
- ☐ E 25

Solution:

The side length is $AD = AP + PD = 5$, so $AB = 5$. In right triangle APB ,

$$\begin{aligned} BP &= \sqrt{AB^2 - AP^2} \\ &= \sqrt{25 - 9} = 4. \end{aligned}$$

Taking AD as the base and BP as the height, the area is $AD \cdot BP = 5 \cdot 4 = 20$.

Thus, the correct answer is **D**.

3. How many of the first ten numbers of the sequence 121, 11211, 1112111, ... are prime numbers?

☒ A 0

☐ B 1

☐ C 2

☐ D 3

☐ E 4

Solution:

The n th term consists of n ones, then a 2, then n ones. It factors as a repunit times a number of the form $10^n + 1$:

$$121 = 11 \cdot 11,$$

$$11211 = 111 \cdot 101,$$

$$1112111 = 1111 \cdot 1001,$$

and in general the n th term equals $\underbrace{1 \cdots 1}_{n+1} \cdot (10^n + 1)$.

For every $n \geq 1$ both factors exceed 1, so every term is composite. None of the ten numbers is prime.

Thus, the correct answer is **A**.

4. For how many values of the constant k will the polynomial $x^2 + kx + 36$ have two distinct integer roots?

A 6

B 8

C 9

D 14

E 16

Solution:

If the roots are integers p and q , then $pq = 36$ and $k = -(p + q)$. Distinct roots must have the same sign, so we list factor pairs of 36 with $p \neq q$.

The positive pairs are $(1, 36)$, $(2, 18)$, $(3, 12)$, $(4, 9)$, and the negative pairs are $(-1, -36)$, $(-2, -18)$, $(-3, -12)$, $(-4, -9)$. The pair $(6, 6)$ is excluded since the roots must be distinct.

Each of these 8 pairs gives a different value of k .

Thus, the correct answer is **B**.

5. The point $(-1, -2)$ is rotated 270° counterclockwise about the point $(3, 1)$. What are the coordinates of its new position?

A $(-3, -4)$

B $(0, 5)$

C $(2, -1)$

D $(4, 3)$

E $(6, -3)$

Solution:

Relative to the center $(3, 1)$, the point is at $(-1 - 3, -2 - 1) = (-4, -3)$.

A 270° counterclockwise rotation sends (x, y) to $(y, -x)$, so $(-4, -3)$ becomes $(-3, 4)$.

Translating back gives $(3 - 3, 1 + 4) = (0, 5)$.

Thus, the correct answer is **B**.

6. Consider the following 100 sets of 10 elements each:

$$\begin{aligned} &\{1, 2, 3, \dots, 10\}, \\ &\{11, 12, 13, \dots, 20\}, \\ &\{21, 22, 23, \dots, 30\}, \\ &\vdots \\ &\{991, 992, 993, \dots, 1000\}. \end{aligned}$$

How many of these sets contain exactly two multiples of 7?

- ☐ A 40
- ☒ B 42
- ☐ C 43
- ☐ D 49
- ☐ E 50

Solution:

Among 1 to 1000 there are $\left\lfloor \frac{1000}{7} \right\rfloor = 142$ multiples of 7. Because $10 > 7$, each block of 10 consecutive integers contains one or two multiples of 7.

If x blocks contain two and the remaining $100 - x$ contain one, then $2x + (100 - x) = 142$, so $x = 42$.

Thus, the correct answer is **B**.

7. Camila writes down five positive integers. The unique mode of these integers is 2 greater than their median, and the median is 2 greater than their arithmetic mean. What is the least possible value for the mode?

- A 5
- B 7
- C 9
- D 11**
- E 13

Solution:

List the numbers in increasing order with median m . The mode is $m + 2 > m$, so it can only occur among the two largest entries; for it to be the unique mode, both of them must equal $m + 2$.

The mean is $m - 2$, so the total is $5(m - 2)$. With the two largest equal to $m + 2$ and the median m , the two smallest sum to $5(m - 2) - m - 2(m + 2) = 2m - 14$.

The two smallest are distinct positive integers, so $2m - 14 \geq 1 + 2 = 3$, giving $m \geq 9$. With $m = 9$ the list 1, 3, 9, 11, 11 works, so the least mode is $m + 2 = 11$.

Thus, the correct answer is **D**.

8. What is the graph of $y^4 + 1 = x^4 + 2y^2$ in the coordinate plane?

- ☐ A two intersecting parabolas
- ☐ B two nonintersecting parabolas
- ☐ C two intersecting circles
- ☒ D a circle and a hyperbola
- ☐ E a circle and two parabolas

Solution:

Rearranging, $y^4 - 2y^2 + 1 = x^4$, so $(y^2 - 1)^2 = (x^2)^2$. This factors as

$$(y^2 - 1 - x^2)(y^2 - 1 + x^2) = 0.$$

Thus either $y^2 - x^2 = 1$, which is a hyperbola, or $x^2 + y^2 = 1$, which is a circle.

Thus, the correct answer is **D**.

9. The sequence $a_0, a_1, a_2, \dots$ is a strictly increasing arithmetic sequence of positive integers such that

$$2^{a_7} = 2^{27} \cdot a_7.$$

What is the minimum possible value of a_2 ?

- ☐ A 8
- ☒ B 12
- ☐ C 16
- ☐ D 17
- ☐ E 22

Solution:

Dividing by 2^{27} , we need $2^{a_7-27} = a_7$. Thus $a_7 = 2^j$ and $2^j - j = 27$, where $j = a_7 - 27 \geq 0$. The left side is strictly increasing for $j \geq 1$, and $j = 5$ works, so uniquely $a_7 = 32$.

With common difference $d \geq 1$, we have $a_7 = a_0 + 7d = 32$ and $a_2 = a_0 + 2d = 32 - 5d$. To minimize a_2 we maximize d ; since $a_0 = 32 - 7d \geq 1$, the largest choice is $d = 4$ (giving $a_0 = 4$).

Then $a_2 = 32 - 20 = 12$.

Thus, the correct answer is **B**.

10. Regular hexagon $ABCDEF$ has side length 2. Let G be the midpoint of $\overline{AB}$, and let H be the midpoint of $\overline{DE}$. What is the perimeter of $GCHF$?

A $4\sqrt{3}$

B 8

C $4\sqrt{5}$

D $4\sqrt{7}$

E 12

Solution:

Place the hexagon with center at the origin: $A = (-1, \sqrt{3})$, $B = (1, \sqrt{3})$, $C = (2, 0)$, $D = (1, -\sqrt{3})$, $E = (-1, -\sqrt{3})$, $F = (-2, 0)$.

Then $G = (0, \sqrt{3})$ and $H = (0, -\sqrt{3})$. By symmetry all four sides of $GCHF$ are equal, and

$$GC = \sqrt{2^2 + (\sqrt{3})^2} = \sqrt{7}.$$

The perimeter is $4\sqrt{7}$.

Thus, the correct answer is **D**.

11. Let

$$f(n) = \left(\frac{-1 + i\sqrt{3}}{2} \right)^n + \left(\frac{-1 - i\sqrt{3}}{2} \right)^n,$$

where $i = \sqrt{-1}$. What is $f(2022)$?

- ☐ A -2
- ☐ B -1
- ☐ C 0
- ☐ D $\sqrt{3}$
- ☒ E 2

Solution:

The two bases are the primitive cube roots of unity, $\omega = e^{\frac{2\pi i}{3}}$ and its conjugate $\omega^2 = e^{-\frac{2\pi i}{3}}$. So $f(n) = \omega^n + \omega^{-n} = 2 \cos \frac{2\pi n}{3}$.

Since 2022 is a multiple of 3, $\omega^{2022} = 1$, so $f(2022) = 1 + 1 = 2$.

Thus, the correct answer is **E**.

12. Kayla rolls four fair 6-sided dice. What is the probability that at least one of the numbers Kayla rolls is greater than 4 and at least two of the numbers she rolls are greater than 2?

A $\frac{2}{3}$

B $\frac{19}{27}$

C $\frac{59}{81}$

D $\frac{61}{81}$

E $\frac{7}{9}$

Solution:

Sort each die into low $\{1, 2\}$, mid $\{3, 4\}$, or high $\{5, 6\}$; each has probability $\frac{1}{3}$, so the $3^4 = 81$ category patterns are equally likely.

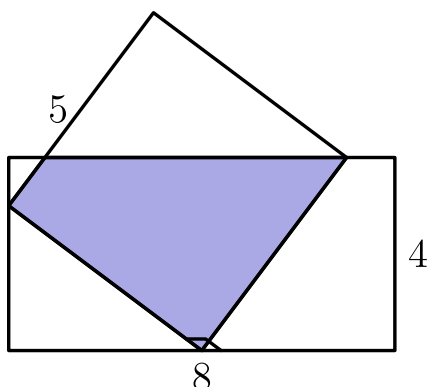
We need at least one high die (a number greater than 4) and at least two dice that are greater than 2 (mid or high). The two bad events are having no high die and having at most one non-low die.

There are $2^4 = 16$ patterns of the first kind, 9 of the second kind, and 5 in their intersection. By inclusion-exclusion the count of good patterns is $81 - 16 - 9 + 5 = 61$.

The probability is $\frac{61}{81}$.

Thus, the correct answer is **D**.

13. The diagram below shows a rectangle with side lengths 4 and 8 and a square with side length 5. Three vertices of the square lie on three different sides of the rectangle, as shown. What is the area of the region inside both the square and the rectangle?



- A $15\frac{1}{8}$
- B $15\frac{3}{8}$
- C $15\frac{1}{2}$
- D $15\frac{5}{8}$**
- E $15\frac{7}{8}$

Solution:

Place the rectangle as $[0, 8] \times [0, 4]$. The tilted square, using the 3-4-5 right triangles, has vertices $(4, 0)$, $(0, 3)$, $(3, 7)$, and $(7, 4)$.

The entire square lies inside the rectangle except for the triangle poking above the top edge $y = 4$. That triangle has vertices $(0.75, 4)$, $(3, 7)$, and $(7, 4)$, with area

$$\frac{1}{2} \cdot (7 - 0.75) \cdot (7 - 4) = \frac{75}{8}.$$

The region inside both is $25 - \frac{75}{8} = \frac{125}{8} = 15\frac{5}{8}$.

Thus, the correct answer is **D**.

14. The graph of $y = x^2 + 2x - 15$ intersects the x -axis at points A and C and the y -axis at point B . What is $\tan(\angle ABC)$?

A $\frac{1}{7}$

B $\frac{1}{4}$

C $\frac{3}{7}$

D $\frac{1}{2}$

E $\frac{4}{7}$

Solution:

Factoring, $x^2 + 2x - 15 = (x + 5)(x - 3)$, so $A = (-5, 0)$ and $C = (3, 0)$, and the y -intercept is $B = (0, -15)$.

Then $\vec{BA} = (-5, 15)$ and $\vec{BC} = (3, 15)$. Using the cross and dot products,

$$\begin{aligned}\tan(\angle ABC) &= \frac{|(-5)(15) - (15)(3)|}{(-5)(3) + (15)(15)} \\ &= \frac{120}{210} \\ &= \frac{4}{7}.\end{aligned}$$

Thus, the correct answer is **E**.

15. One of the following numbers is not divisible by any prime number less than 10. Which is it?

A $2^{606} - 1$

B $2^{606} + 1$

C $2^{607} - 1$

D $2^{607} + 1$

E $2^{607} + 3^{607}$

Solution:

Every option is odd, so only the primes 3, 5, 7 need checking.

Option A: $2^{606} \equiv 1 \pmod{3}$, so $2^{606} - 1$ is divisible by 3. Option B: $2^{606} \equiv 4 \pmod{5}$, so $2^{606} + 1$ is divisible by 5. Option D: $2^{607} \equiv 2 \pmod{3}$, so $2^{607} + 1$ is divisible by 3. Option E: modulo 5, $2^{607} + 3^{607} \equiv 3 + 2 = 5 \equiv 0$.

For $2^{607} - 1$: it is $\equiv 1 \pmod{3}$, $\equiv 2 \pmod{5}$, and (since $2^3 \equiv 1 \pmod{7}$ and $607 \equiv 1 \pmod{3}$) $\equiv 1 \pmod{7}$. So it is not divisible by any prime below 10.

Thus, the correct answer is **C**.

16. Suppose x and y are positive real numbers such that

$$x^y = 2^{64}$$

and

$$(\log_2 x)^{\log_2 y} = 2^7.$$

What is the greatest possible value of $\log_2 y$?

A 3

B 4

C $3 + \sqrt{2}$

D $4 + \sqrt{3}$

E 7

Solution:

Let $a = \log_2 x$ and $b = \log_2 y$. Taking $\log_2$ of $x^y = 2^{64}$ gives $y \log_2 x = 64$, i.e. $a \cdot 2^b = 2^6$.

Taking $\log_2$ of the second equation gives $b \log_2 a = 7$, so $a = 2^{\frac{7}{b}}$. Substituting, $2^{\frac{7}{b}} \cdot 2^b = 2^6$, so $b + \frac{7}{b} = 6$, i.e. $b^2 - 6b + 7 = 0$.

Thus $b = 3 \pm \sqrt{2}$, and the greatest value of $\log_2 y$ is $3 + \sqrt{2}$.

Thus, the correct answer is **C**.

17. How many 4×4 arrays whose entries are 0s and 1s are there such that the row sums (the sum of the entries in each row) are 1, 2, 3, and 4, in some order, and the column sums (the sum of the entries in each column) are also 1, 2, 3, and 4, in some order? For example, the array

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

satisfies the condition.

- A 144
- B 240
- C 336
- D 576**
- E 624

Solution:

The row with sum 4 is all 1s and the column with sum 4 is all 1s. There are $4!$ ways to assign the row sums 1, 2, 3, 4 to the four rows, and 4 choices for which column has sum 4.

Delete that column. The remaining 4×3 array has row sums 0, 1, 2, 3 and must have column sums 1, 2, 3. The all-zero and all-one rows are forced; the rows of reduced sum 1 and 2 can be placed in 6 ways to produce column sums 1, 2, 3 in some order.

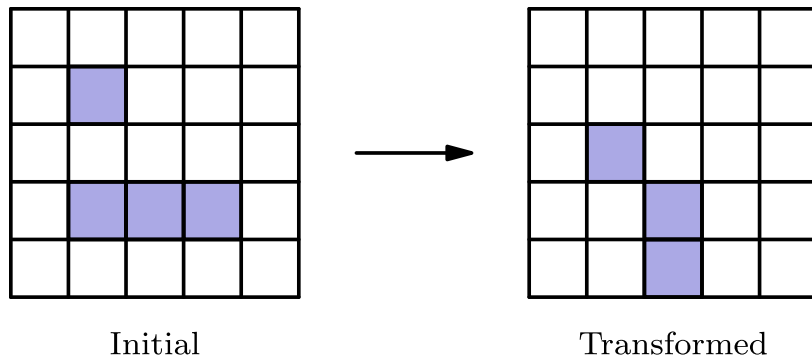
The total is $24 \cdot 4 \cdot 6 = 576$.

Thus, the correct answer is **D**.

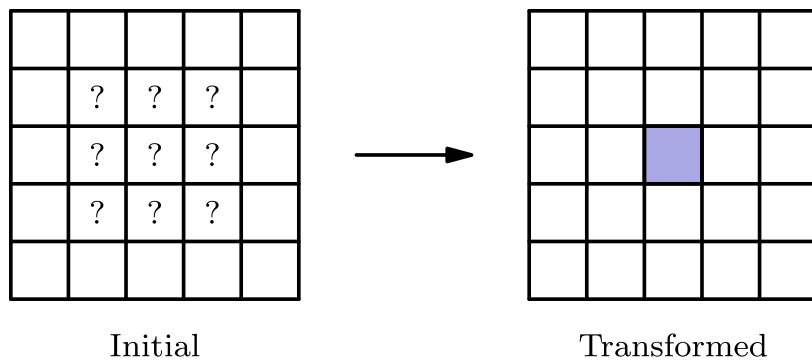
18. Each square in a 5×5 grid is either filled or empty, and has up to eight adjacent neighboring squares, where neighboring squares share either a side or a corner. The grid is transformed by the following rules:

Any filled square with two or three filled neighbors remains filled. Any empty square with exactly three filled neighbors becomes a filled square. All other squares remain empty or become empty.

A sample transformation is shown in the figure below.



Suppose the 5×5 grid has a border of empty squares surrounding a 3×3 subgrid. How many initial configurations will lead to a transformed grid consisting of a single filled square in the center after a single transformation? (Rotations and reflections of the same configuration are considered different.)



- A 14
- B 18
- C 22
- D 26

Solution:

Only the inner 3×3 squares can start filled. For the center to be filled afterward, if it began empty it needs exactly 3 filled neighbors, and if it began filled it needs 2 or 3.

Every other square must end empty. The key restriction is that no border square may acquire exactly three filled neighbors, which rules out filling all three squares along an outer edge of the 3×3 .

If the center starts empty, exactly 3 of its 8 neighbors must be filled. Checking these triples up to square symmetry leaves four types. With ring coordinates centered at $(0, 0)$, representatives are

$$\begin{aligned} &\{(-1, -1), (-1, 0), (1, 1)\}, \\ &\{(-1, -1), (-1, 1), (1, -1)\}, \\ &\{(-1, -1), (-1, 1), (1, 0)\}, \\ &\{(-1, -1), (0, 1), (1, 0)\}. \end{aligned}$$

Their symmetry-orbit sizes are 8, 4, 4, 4, giving 20 configurations.

If the center starts filled, the same neighbor check leaves only the two configurations in which the other filled cells are opposite corner neighbors. Hence the total is $20 + 2 = 22$.

Thus, the correct answer is **C**.

19. In $\triangle ABC$ medians $\overline{AD}$ and $\overline{BE}$ intersect at G and $\triangle AGE$ is equilateral. Then $\cos(C)$ can be written as $\frac{m\sqrt{p}}{n}$, where m and n are relatively prime positive integers and p is a positive integer not divisible by the square of any prime. What is $m + n + p$?

A 44

B 48

C 52

D 56

E 60

Solution:

Let $a = BC$, $b = CA$, $c = AB$. Since E is the midpoint of AC , $AE = \frac{b}{2}$. The centroid gives $AG = \frac{2}{3}m_a$ and $GE = \frac{1}{3}m_b$, where m_a , m_b are the medians from A and B .

Equilateral $\triangle AGE$ means $AG = GE = AE$. From $\frac{2}{3}m_a = \frac{b}{2}$ we get $m_a = \frac{3}{4}b$, which with $m_a^2 = \frac{2b^2 + 2c^2 - a^2}{4}$ gives $2c^2 - a^2 = \frac{b^2}{4}$. From $\frac{1}{3}m_b = \frac{b}{2}$ we get $m_b = \frac{3}{2}b$, giving $a^2 + c^2 = 5b^2$.

Solving, $c^2 = \frac{7b^2}{4}$ and $a^2 = \frac{13b^2}{4}$. Taking $b = 2$ gives $a^2 = 13$, $c^2 = 7$, so

$$\begin{aligned}\cos C &= \frac{a^2 + b^2 - c^2}{2ab} \\ &= \frac{13 + 4 - 7}{2 \cdot \sqrt{13} \cdot 2} \\ &= \frac{5\sqrt{13}}{26}.\end{aligned}$$

Then $m + n + p = 5 + 26 + 13 = 44$.

Thus, the correct answer is **A**.

20. Let $P(x)$ be a polynomial with rational coefficients such that when $P(x)$ is divided by the polynomial $x^2 + x + 1$, the remainder is $x + 2$, and when $P(x)$ is divided by the polynomial $x^2 + 1$, the remainder is $2x + 1$. There is a unique polynomial of least degree with these two properties. What is the sum of the squares of the coefficients of that polynomial?

A 10

B 13

C 19

D 20

E 23

Solution:

No linear polynomial can have the two different remainders. If $P(x) = ax^2 + bx + c$ were quadratic, comparing its remainders would give both $b - a = 1$, $c - a = 2$ and $b = 2$, $c - a = 1$, a contradiction. Thus the least possible degree is 3. Write $P(x) = (x + 2) + (x^2 + x + 1)(px + q)$, which has remainder $x + 2$ upon division by $x^2 + x + 1$.

Reducing modulo $x^2 + 1$ (so $x^2 \equiv -1$) gives remainder $(q + 1)x + (2 - p)$. Setting this equal to $2x + 1$ gives $q = 1$ and $p = 1$.

Then $P(x) = x^3 + 2x^2 + 3x + 3$, and the sum of the squares of the coefficients is $1 + 4 + 9 + 9 = 23$.

Thus, the correct answer is **E**.

21. Let S be the set of circles in the coordinate plane that are tangent to each of the three circles with equations $x^2 + y^2 = 4$, $x^2 + y^2 = 64$, and $(x - 5)^2 + y^2 = 3$. What is the sum of the areas of all circles in S ?

A 48π

B 68π

C 96π

D 102π

E 136π

Solution:

The first two circles are concentric with radii 2 and 8. A circle tangent to both either has radius 3 with center at distance 5 from the origin, or radius 5 with center at distance 3 from the origin.

The third circle has center $(5, 0)$ and radius $\sqrt{3}$. For each candidate radius $s \in \{3, 5\}$, tangency requires the center's distance from $(5, 0)$ to be $s + \sqrt{3}$ or $s - \sqrt{3}$. Each of these two distance circles intersects the appropriate center-locus in two symmetric points. Hence exactly four radius-3 circles and four radius-5 circles work.

The sum of the areas is $4 \cdot \pi(3)^2 + 4 \cdot \pi(5)^2 = 36\pi + 100\pi = 136\pi$.

Thus, the correct answer is **E**.

22. Ant Amelia starts on the number line at 0 and crawls in the following manner. For $n = 1, 2, 3$, Amelia chooses a time duration t_n and an increment x_n independently and uniformly at random from the interval $(0, 1)$. During the n th step of the process, Amelia moves x_n units in the positive direction, using up t_n minutes. If the total elapsed time has exceeded 1 minute during the n th step, she stops at the end of that step; otherwise, she continues with the next step, taking at most 3 steps in all. What is the probability that Amelia's position when she stops will be greater than 1?

A $\frac{1}{3}$

B $\frac{1}{2}$

C $\frac{2}{3}$

D $\frac{3}{4}$

E $\frac{5}{6}$

Solution:

Because each $t_n < 1$, Amelia always completes at least two steps. She stops after exactly two steps when $t_1 + t_2 > 1$, which happens with probability $\frac{1}{2}$; otherwise she takes all three steps.

The increments are independent of the times. If she takes two steps, her position is $x_1 + x_2$, and $P(x_1 + x_2 > 1) = \frac{1}{2}$. If she takes three, her position is $x_1 + x_2 + x_3$, and $P(x_1 + x_2 + x_3 > 1) = 1 - \frac{1}{6} = \frac{5}{6}$.

The answer is $\frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{5}{6} = \frac{1}{4} + \frac{5}{12} = \frac{2}{3}$.

Thus, the correct answer is **C**.

23. Let $x_0, x_1, x_2, \dots$ be a sequence of numbers, where each x_k is either 0 or 1. For each positive integer n , define

$$S_n = \sum_{k=0}^{n-1} x_k 2^k.$$

Suppose $7S_n \equiv 1 \pmod{2^n}$ for all $n \geq 1$. What is the value of the sum

$$x_{2019} + 2x_{2020} + 4x_{2021} + 8x_{2022}?$$

A 6

B 7

C 12

D 14

E 15

Solution:

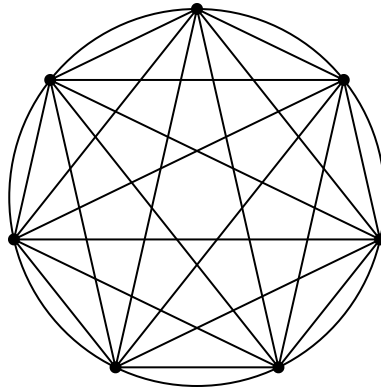
Since S_n is the integer formed by the low n bits, the condition $7S_n \equiv 1 \pmod{2^n}$ means $S_n \equiv 7^{-1} \pmod{2^n}$ for every n . Thus the digits x_k are the base-2 digits of $\frac{1}{7}$ as a 2-adic number.

Long division in base 2 gives digits $x_0 = x_1 = x_2 = 1$, and thereafter the block repeats with period 3 : for $k \geq 1$, $x_k = 0$ exactly when $3 \mid k$, and $x_k = 1$ otherwise.

Since $3 \mid 2019$ and $3 \mid 2022$, while $2020 \equiv 1$ and $2021 \equiv 2 \pmod{3}$, we get $x_{2019} = 0, x_{2020} = 1, x_{2021} = 1, x_{2022} = 0$. The sum is $0 + 2 + 4 + 0 = 6$.

Thus, the correct answer is **A**.

24. The figure below depicts a regular 7-gon inscribed in a unit circle.



What is the sum of the 4th powers of the lengths of all 21 of its edges and diagonals?

A 49

B 98

C 147

D 168

E 196

Solution:

A chord joining two vertices d steps apart has squared length $2 - 2 \cos \frac{2\pi d}{7}$, and there are 7 chords for each of $d = 1, 2, 3$. The required sum is

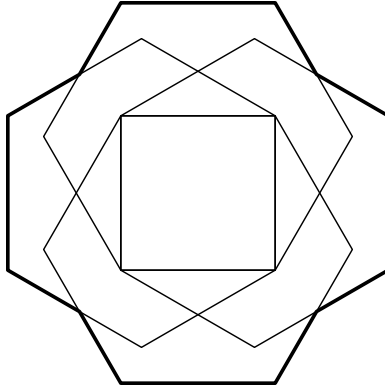
$$7 \sum_{d=1}^3 \left(2 - 2 \cos \frac{2\pi d}{7}\right)^2.$$

Using $\sum_{d=1}^3 \cos \frac{2\pi d}{7} = -\frac{1}{2}$ and $\sum_{d=1}^3 \cos^2 \frac{2\pi d}{7} = \frac{5}{4}$, the inner sum expands to $4 \left(3 + 1 + \frac{5}{4}\right) = 21$.

Therefore the total is $7 \cdot 21 = 147$.

Thus, the correct answer is **C**.

25. Four regular hexagons surround a square with a side length 1, each one sharing an edge with the square, as shown in the figure below. The area of the resulting 12-sided outer nonconvex polygon can be written as $m\sqrt{n} + p$, where m , n , and p are integers and n is not divisible by the square of any prime. What is $m + n + p$?



A -12

B -4

C 4

D 24

E 32

Solution:

Center the square at the origin with vertices $(\pm\frac{1}{2}, \pm\frac{1}{2})$. Each hexagon shares one edge with the square and extends across to the opposite side; the hexagon on the bottom edge, for instance, has its far (top) edge from $(-\frac{1}{2}, \sqrt{3} - \frac{1}{2})$ to $(\frac{1}{2}, \sqrt{3} - \frac{1}{2})$.

The outer boundary is a 12-gon with flat edges at distance $\sqrt{3} - \frac{1}{2}$ from the center, convex vertices such as $(\sqrt{3} - \frac{1}{2}, \frac{1}{2})$, and four reflex notches where adjacent hexagons' slanted edges meet, at $(\frac{5}{2} - \sqrt{3}, \frac{5}{2} - \sqrt{3})$ and its symmetric images.

Applying the shoelace formula to these 12 vertices gives area $16\sqrt{3} - 23$, so $m = 16$, $n = 3$, $p = -23$, and $m + n + p = -4$.

Thus, the correct answer is **B**.

Problems: <https://live.poshenloh.com/past-contests/amc12/2022B>

