

AMC 12 SOLUTION ARCHIVE

2021

FALL 2021

12A and 12B full explanations

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2021 AMC 12A Fall Solutions

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1. What is the value of

$$\frac{(2112 - 2021)^2}{169}?$$

A 7

B 21

C 49

D 64

E 91

Solution:

Since $2112 - 2021 = 91 = 7 \cdot 13$ and $169 = 13^2$, the fraction is $\frac{(7 \cdot 13)^2}{13^2} = 7^2 = 49$.

Thus, the correct answer is **C**.

2. Menkara has a 4×6 index card. If she shortens the length of one side of this card by 1 inch, the card would have area 18 square inches. What would the area of the card be in square inches if instead she shortens the length of the other side by 1 inch?

A 16

B 17

C 18

D 19

E 20

Solution:

The original card is 4×6 . Shortening a side by 1 inch gives area 18, which requires $3 \times 6 = 18$, so the reduced side was the 4-inch side.

Shortening the other side by 1 inch instead gives $4 \times 5 = 20$ square inches.

Thus, the correct answer is **E**.

3. Mr. Lopez has a choice of two routes to get to work. Route A is 6 miles long, and his average speed along this route is 30 miles per hour. Route B is 5 miles long, and his average speed along this route is 40 miles per hour, except for a $\frac{1}{2}$ -mile stretch in a school zone where his average speed is 20 miles per hour. By how many minutes is Route B quicker than Route A?

A $2\frac{3}{4}$

B $3\frac{3}{4}$

C $4\frac{1}{2}$

D $5\frac{1}{2}$

E $6\frac{3}{4}$

Solution:

Route A takes $\frac{6}{30} = \frac{1}{5}$ hour = 12 minutes.

Route B has 4.5 miles at 40 mph and 0.5 mile at 20 mph, taking $\frac{4.5}{40} + \frac{0.5}{20} = 0.1125 + 0.025 = 0.1375$ hour = 8.25 minutes.

The difference is $12 - 8.25 = 3.75 = 3\frac{3}{4}$ minutes.

Thus, the correct answer is **B**.

4. The six-digit number $\underline{2}\underline{0}\underline{2}\underline{1}\underline{0}\underline{A}$ is prime for only one digit A . What is A ?

- ☐ A 1
- ☐ B 3
- ☐ C 5
- ☐ D 7
- ☒ E 9

Solution:

The number is $202100 + A$. Any even A makes it even, and $A = 5$ makes it divisible by 5, so A must be odd and not 5.

For $A = 1$ the digit sum is 6 (divisible by 3); for $A = 7$ the digit sum is 12 (divisible by 3); and $202103 = 11 \cdot 18373$. Only 202109 survives all tests, and it is prime.

Thus, the correct answer is **E**.

5. Elmer the emu takes 44 equal strides to walk between consecutive telephone poles on a rural road. Oscar the ostrich can cover the same distance in 12 equal leaps. The telephone poles are evenly spaced, and the 41st pole along this road is exactly one mile (5280 feet) from the first pole. How much longer, in feet, is Oscar's leap than Elmer's stride?

A 6

B 8

C 10

D 11

E 15

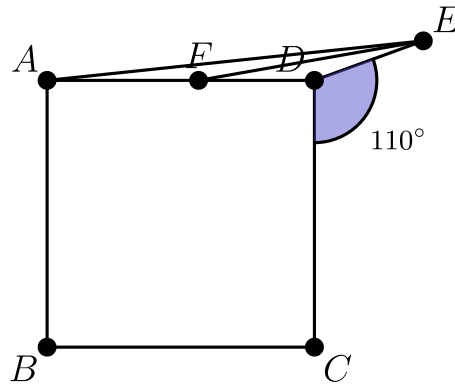
Solution:

There are 40 gaps between the first and 41st poles, so each gap is $\frac{5280}{40} = 132$ feet.

Elmer's stride is $\frac{132}{44} = 3$ feet and Oscar's leap is $\frac{132}{12} = 11$ feet, a difference of $11 - 3 = 8$ feet.

Thus, the correct answer is **B**.

6. As shown in the figure below, point E lies on the opposite half-plane determined by line CD from point A so that $\angle CDE = 110^\circ$. Point F lies on $\overline{AD}$ so that $DE = DF$, and $ABCD$ is a square. What is the degree measure of $\angle AFE$?



- ☐ A 160
- ☐ B 164
- ☐ C 166
- ☒ D 170
- ☐ E 174

Solution:

Because $ABCD$ is a square, $\angle ADC = 90^\circ$. Since E and A lie on opposite sides of line CD , ray DE is swung past DC , so the angle of triangle DFE at D (with F on $\overline{AD}$) is $\angle FDE = 360^\circ - (\angle ADC + \angle CDE) = 360^\circ - (90^\circ + 110^\circ) = 160^\circ$.

Since $DF = DE$, triangle DFE is isosceles with base angles $\angle DFE = \frac{180^\circ - 160^\circ}{2} = 10^\circ$.

As A, F, D are collinear, $\angle AFE = 180^\circ - \angle DFE = 170^\circ$.

Thus, the correct answer is **D**.

7. A school has 100 students and 5 teachers. In the first period, each student is taking one class, and each teacher is teaching one class. The enrollments in the classes are 50, 20, 20, 5, and 5. Let t be the average value obtained if a teacher is picked at random and the number of students in their class is noted. Let s be the average value obtained if a student was picked at random and the number of students in their class, including the student, is noted. What is $t - s$?

A -18.5

B -13.5

C 0

D 13.5

E 18.5

Solution:

The teacher average is $t = \frac{50 + 20 + 20 + 5 + 5}{5} = \frac{100}{5} = 20$.

The student average weights each class size by how many students are in it:

$$\begin{aligned} s &= \frac{50^2 + 20^2 + 20^2 + 5^2 + 5^2}{100} \\ &= \frac{2500 + 400 + 400 + 25 + 25}{100} \\ &= 33.5. \end{aligned}$$

So $t - s = 20 - 33.5 = -13.5$.

Thus, the correct answer is **B**.

8. Let M be the least common multiple of all the integers 10 through 30, inclusive. Let N be the least common multiple of $M, 32, 33, 34, 35, 36, 37, 38, 39$, and 40. What is the value of $\frac{N}{M}$?

- A 1
- B 2
- C 37
- D 74**
- E 2886

Solution:

$M = \text{lcm}(10, \dots, 30)$ contains 2^4 (from 16), 3^3 (from 27), 5^2 (from 25), 7, and every prime up to 29.

Among 32, $\dots$, 40, the only new contributions are $32 = 2^5$, which raises the power of 2 from 2^4 to 2^5 , and the new prime 37. Everything else factors into primes and powers already in M .

Therefore $\frac{N}{M} = 2 \cdot 37 = 74$.

Thus, the correct answer is **D**.

9. A right rectangular prism whose surface area and volume are numerically equal has edge lengths $\log_2 x$, $\log_3 x$, and $\log_4 x$. What is x ?

A $2\sqrt{6}$

B $6\sqrt{6}$

C 24

D 48

E 576

Solution:

Let $a = \log_2 x$, $b = \log_3 x$, $c = \log_4 x$. Surface area equals volume gives $2(ab + bc + ca) = abc$. Dividing by abc ,

$$1 = 2 \left(\frac{1}{c} + \frac{1}{a} + \frac{1}{b} \right).$$

Since $\frac{1}{a} = \log_x 2$, etc., the sum is $\log_x 2 + \log_x 3 + \log_x 4 = \log_x 24$. Thus $1 = 2 \log_x 24$, so $\log_x 24 = \frac{1}{2}$, meaning $x^{\frac{1}{2}} = 24$ and $x = 576$.

Thus, the correct answer is **E**.

10. The base-nine representation of the number N is $27,006,000,052_{\text{nine}}$. What is the remainder when N is divided by 5?

- ☐ A 0
- ☐ B 1
- ☐ C 2
- ☒ D 3
- ☐ E 4

Solution:

Since $9 \equiv -1 \pmod{5}$, each power $9^k \equiv (-1)^k$, so N is congruent to the alternating sum of its base-nine digits.

The nonzero digits, with their positions from the right, are 2 (position 0), 5 (position 1), 6 (position 6), 7 (position 9), and 2 (position 10). The alternating sum is $2 - 5 + 6 - 7 + 2 = -2 \equiv 3 \pmod{5}$.

Thus, the correct answer is **D**.

11. Consider two concentric circles of radius 17 and 19. The larger circle has a chord, half of which lies inside the smaller circle. What is the length of the chord in the larger circle?

A $12\sqrt{2}$

B $10\sqrt{3}$

C $\sqrt{17 \cdot 19}$

D 18

E $8\sqrt{6}$

Solution:

Let the chord lie at distance d from the common center. Its total length is $2\sqrt{361 - d^2}$, and the portion inside the smaller circle has length $2\sqrt{289 - d^2}$.

Since half the chord lies inside, $2\sqrt{289 - d^2} = \frac{1}{2} \cdot 2\sqrt{361 - d^2}$. Squaring gives $4(289 - d^2) = 361 - d^2$, so $3d^2 = 795$ and $d^2 = 265$.

The chord length is $2\sqrt{361 - 265} = 2\sqrt{96} = 8\sqrt{6}$.

Thus, the correct answer is **E**.

12. What is the number of terms with rational coefficients among the 1001 terms in the expansion of

$$\left(x\sqrt[3]{2} + y\sqrt{3}\right)^{1000}?$$

- A 0
- B 166
- C 167**
- D 500
- E 501

Solution:

The general term is $\binom{1000}{k} (x\sqrt[3]{2})^{1000-k} (y\sqrt{3})^k$, whose coefficient contains $2^{\frac{1000-k}{3}}$ and $3^{\frac{k}{2}}$. This is rational exactly when $3 \mid (1000 - k)$ and k is even.

Since $1000 \equiv 1 \pmod{3}$, we need $k \equiv 1 \pmod{3}$ and k even, which combine to $k \equiv 4 \pmod{6}$. The valid values $k = 4, 10, \dots, 1000$ number $\frac{1000 - 4}{6} + 1 = 167$.

Thus, the correct answer is **C**.

13. The angle bisector of the acute angle formed at the origin by the graphs of the lines $y = x$ and $y = 3x$ has equation $y = kx$. What is k ?

A $\frac{1 + \sqrt{5}}{2}$

B $\frac{1 + \sqrt{7}}{2}$

C $\frac{2 + \sqrt{3}}{2}$

D 2

E $\frac{2 + \sqrt{5}}{2}$

Solution:

The bisector points along the sum of the unit vectors of the two lines: $\frac{(1, 1)}{\sqrt{2}} + \frac{(1, 3)}{\sqrt{10}}$.

Its slope is

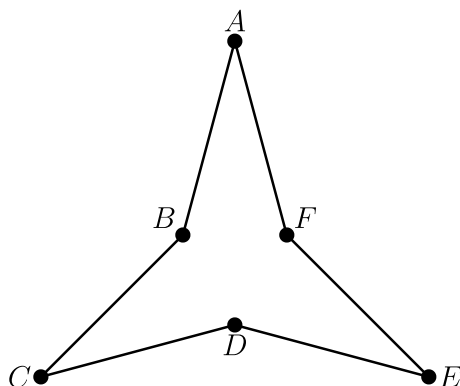
$$k = \frac{\frac{1}{\sqrt{2}} + \frac{3}{\sqrt{10}}}{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{10}}} = \frac{\sqrt{5} + 3}{\sqrt{5} + 1}.$$

Multiplying numerator and denominator by $\sqrt{5} - 1$ gives $\frac{(\sqrt{5} + 3)(\sqrt{5} - 1)}{4} =$

$$\frac{2 + 2\sqrt{5}}{4} = \frac{1 + \sqrt{5}}{2}.$$

Thus, the correct answer is **A**.

14. In the figure, equilateral hexagon $ABCDEF$ has three nonadjacent acute interior angles that each measure 30° . The enclosed area of the hexagon is $6\sqrt{3}$. What is the perimeter of the hexagon?



- ☐ A 4
- ☐ B $4\sqrt{3}$
- ☐ C 12
- ☐ D 18
- ☒ E $12\sqrt{3}$

Solution:

Let the common side length be s . The three acute vertices are the tips of isosceles triangles with two sides s and apex 30° ; each has area $\frac{1}{2}s^2 \sin 30^\circ = \frac{s^2}{4}$.

The three reflex vertices form an inner equilateral triangle with side $2s \sin 15^\circ$, whose area is $\sqrt{3}s^2 \sin^2 15^\circ$. Using $\sin^2 15^\circ = \frac{2-\sqrt{3}}{4}$, the total area is

$$\frac{3s^2}{4} + \sqrt{3}s^2 \cdot \frac{2-\sqrt{3}}{4} = \frac{s^2\sqrt{3}}{2}.$$

Setting $\frac{s^2\sqrt{3}}{2} = 6\sqrt{3}$ gives $s^2 = 12$, so $s = 2\sqrt{3}$ and the perimeter is $6s = 12\sqrt{3}$.

Thus, the correct answer is **E**.

15. Recall that the conjugate of the complex number $w = a + bi$, where a and b are real numbers and $i = \sqrt{-1}$, is the complex number $\bar{w} = a - bi$. For any complex number z , let $f(z) = 4i\bar{z}$. The polynomial

$$P(z) = z^4 + 4z^3 + 3z^2 + 2z + 1$$

has four complex roots: z_1, z_2, z_3 , and z_4 . Let

$$Q(z) = z^4 + Az^3 + Bz^2 + Cz + D$$

be the polynomial whose roots are $f(z_1), f(z_2), f(z_3)$, and $f(z_4)$, where the coefficients A, B, C , and D are complex numbers. What is $B + D$?

A -304

B -208

C $12i$

D 208

E 304

Solution:

By Vieta on P , $\sum_{i < j} z_i z_j = 3$ and $\prod z_j = 1$, both real, so their conjugates are also 3 and 1.

The roots of Q are $4i\bar{z}_j$. Then B is the sum of products of pairs: $B = (4i)^2 \sum_{i < j} \bar{z}_i \bar{z}_j = -16 \cdot 3 = -48$. And $D = (4i)^4 \prod \bar{z}_j = 256 \cdot 1 = 256$.

So $B + D = -48 + 256 = 208$.

Thus, the correct answer is **D**.

16. An organization has 30 employees, 20 of whom have a brand A computer while the other 10 have a brand B computer. For security, the computers can only be connected to each other and only by cables. The cables can only connect a brand A computer to a brand B computer. Employees can communicate with each other if their computers are directly connected by a cable or by relaying messages through a series of connected computers. Initially, no computer is connected to any other. A technician arbitrarily selects one computer of each brand and installs a cable between them, provided there is not already a cable between that pair. The technician stops once every employee can communicate with each other. What is the maximum possible number of cables used?

A 190

B 191

C 192

D 195

E 196

Solution:

The technician keeps adding cables until the graph becomes connected. To maximize the count, keep the network disconnected for as long as possible: leave a single brand A computer isolated and fully connect the remaining 19 brand A computers to all 10 brand B computers.

That uses $19 \cdot 10 = 190$ cables while still disconnected. The next cable connects the last brand A computer, joining everyone, for a total of $190 + 1 = 191$.

Thus, the correct answer is **B**.

17. For how many ordered pairs (b, c) of positive integers does neither $x^2 + bx + c = 0$ nor $x^2 + cx + b = 0$ have two distinct real solutions?

- A 4
- B 6**
- C 8
- D 12
- E 16

Solution:

Neither quadratic has two distinct real roots exactly when both discriminants are nonpositive: $b^2 \leq 4c$ and $c^2 \leq 4b$.

Combining $c \geq \frac{b^2}{4}$ with $c \leq 2\sqrt{b}$ gives $b^{\frac{3}{2}} \leq 8$, so $b \leq 4$. Checking: $b = 1$ gives $c \in \{1, 2\}$; $b = 2$ gives $c \in \{1, 2\}$; $b = 3$ gives $c = 3$; and $b = 4$ gives $c = 4$.

That is $(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)$ — 6 ordered pairs.

Thus, the correct answer is **B**.

18. Each of 20 balls is tossed independently and at random into one of 5 bins. Let p be the probability that some bin ends up with 3 balls, another with 5 balls, and the other three with 4 balls each. Let q be the probability that every bin ends up with 4 balls. What is $\frac{p}{q}$?

- A 1
- B 4
- C 8
- D 12
- E 16

Solution:

Both probabilities divide by 5^{20} , so $\frac{p}{q}$ is a ratio of arrangement counts.

For q , all bins have 4 : $\frac{20!}{(4!)^5}$. For p , choose which bin has 3 and which has 5 in $5 \cdot 4 = 20$ ways, times $\frac{20!}{3! 5! (4!)^3}$. Therefore

$$\begin{aligned}\frac{p}{q} &= 20 \cdot \frac{(4!)^5}{3! 5! (4!)^3} \\ &= 20 \cdot \frac{(4!)^2}{3! 5!} \\ &= 20 \cdot \frac{576}{720} \\ &= 16.\end{aligned}$$

Thus, the correct answer is **E**.

19. Let x be the least real number greater than 1 such that $\sin(x) = \sin(x^2)$, where the arguments are in degrees. What is x rounded up to the closest integer?

A 10

B 13

C 14

D 19

E 20

Solution:

Equal sines require $x^2 = x + 360k$ or $x^2 = 180 - x + 360k$ for some integer k .

The family $x^2 = x + 360k$ first exceeds 1 at $k = 1$, giving $x \approx 19.5$. The family $x^2 = 180 - x + 360k$ with $k = 0$ gives $x^2 + x - 180 = 0$, so $x = \frac{-1 + \sqrt{721}}{2} \approx 12.93$, which is smaller.

Rounded up, $x = 13$.

Thus, the correct answer is **B**.

20. For each positive integer n , let $f_1(n)$ be twice the number of positive integer divisors of n , and for $j \geq 2$, let $f_j(n) = f_1(f_{j-1}(n))$. For how many values of $n \leq 50$ is $f_{50}(n) = 12$?

- ☐ A 7
- ☐ B 8
- ☐ C 9
- ☒ D 10
- ☐ E 11

Solution:

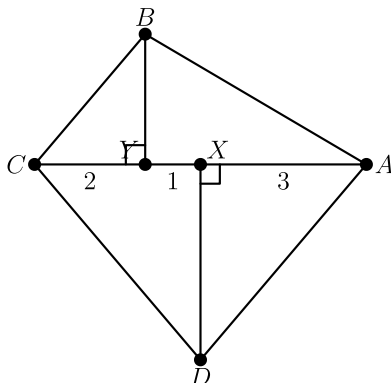
Both 8 and 12 are fixed. For $n \leq 50$, the first value $f_1(n) = 2d(n)$ is at most 20.

Checking the even values through 20 shows that an orbit reaches 12 exactly when its first value is 12, 18, or 20. Thus we need $d(n) = 6, 9$, or 10.

The numbers at most 50 with 6 divisors are 12, 18, 20, 28, 32, 44, 45, 50; the only one with 9 divisors is 36, and the only one with 10 divisors is 48. These 10 values all reach the fixed point 12.

Thus, the correct answer is **D**.

21. Let $ABCD$ be an isosceles trapezoid with $\overline{BC} \parallel \overline{AD}$ and $AB = CD$. Points X and Y lie on diagonal $\overline{AC}$ with X between A and Y , as shown in the figure. Suppose $\angle AXD = \angle BYC = 90^\circ$, $AX = 3$, $XY = 1$, and $YC = 2$. What is the area of $ABCD$?



- ☐ A 15
- ☐ B $5\sqrt{11}$
- ☒ C $3\sqrt{35}$
- ☐ D 18
- ☐ E $7\sqrt{7}$

Solution:

Put $A = (0, 0)$, $X = (3, 0)$, $Y = (4, 0)$, $C = (6, 0)$. The right angles give $D = (3, t)$ and $B = (4, s)$ on opposite sides of AC .

Parallelism $\overline{AD} \parallel \overline{BC}$ forces $t = -\frac{3}{2}s$, and $AB = CD$ gives $16 + s^2 = 9 + t^2$, so $t^2 - s^2 = 7$. Substituting yields $s^2 = \frac{28}{5}$.

The shoelace formula gives area $= 3|t - s| = 3 \cdot \frac{5}{2}s = \frac{15}{2}s = \frac{15}{2}\sqrt{\frac{28}{5}} = 3\sqrt{35}$.

Thus, the correct answer is **C**.

22. Azar and Carl play a game of tic-tac-toe. Azar places an X in one of the boxes in a 3-by-3 array of boxes, then Carl places an O in one of the remaining boxes. After that, Azar places an X in one of the remaining boxes, and so on until all 9 boxes are filled or one of the players has 3 of their symbols in a row — horizontal, vertical, or diagonal — whichever comes first, in which case that player wins the game. Suppose the players make their moves at random, rather than trying to follow a rational strategy, and that Carl wins the game when he places his third O . How many ways can the board look after the game is over?

A 36

B 112

C 120

D 148

E 160

Solution:

Carl wins on his third O , so the board has three O s forming one of the 8 lines and three X s in the other six cells. The X s must not form a line (else Azar would have won first).

If the O line is a row or column (6 choices), the remaining six cells contain two full lines, so valid X placements number $\binom{6}{3} - 2 = 18$. If the O line is a diagonal (2 choices), the remaining six cells contain no full line, giving $\binom{6}{3} = 20$.

The total is $6 \cdot 18 + 2 \cdot 20 = 108 + 40 = 148$.

Thus, the correct answer is **D**.

23. A quadratic polynomial with real coefficients and leading coefficient 1 is called *disrespectful* if the equation $p(p(x)) = 0$ is satisfied by exactly three real numbers. Among all the disrespectful quadratic polynomials, there is a unique such polynomial $\tilde{p}(x)$ for which the sum of the roots is maximized. What is $\tilde{p}(1)$?

A $\frac{5}{16}$

B $\frac{1}{2}$

C $\frac{5}{8}$

D 1

E $\frac{9}{8}$

Solution:

Let p have roots r and s . Then $p(p(x)) = 0$ splits into $p(x) = r$ and $p(x) = s$, with discriminants $(r - s)^2 + 4r$ and $(r - s)^2 + 4s$. Exactly three real roots means one discriminant is 0 and the other positive.

Take $(r - s)^2 + 4s = 0$ and set $u = r - s$. Then $s = -\frac{u^2}{4}$ and the other discriminant is $4u$, so $u > 0$. Also $r + s = -\frac{u^2}{2} + u$, maximized at $u = 1$, giving $r = \frac{3}{4}$ and $s = -\frac{1}{4}$.

So $\tilde{p}(x) = (x - \frac{3}{4})(x + \frac{1}{4})$, and $\tilde{p}(1) = \frac{1}{4} \cdot \frac{5}{4} = \frac{5}{16}$.

Thus, the correct answer is **A**.

24. Convex quadrilateral $ABCD$ has $AB = 18$, $\angle A = 60^\circ$, and $\overline{AB} \parallel \overline{CD}$. In some order, the lengths of the four sides form an arithmetic progression, and side AB is a side of maximum length. The length of another side is a . What is the sum of all possible values of a ?

A 24

B 42

C 60

D 66

E 84

Solution:

Since $AB = 18$ is the largest, the four sides are $18, 18 - d, 18 - 2d, 18 - 3d$.

Placing $A = (0, 0)$, $B = (18, 0)$, and $D = \left(\frac{m}{2}, \frac{m\sqrt{3}}{2}\right)$ with $m = DA$, let $n = CD$ and $\ell = BC$. Then $C = \left(\frac{m}{2} + n, \frac{m\sqrt{3}}{2}\right)$, so

$$\begin{aligned}\ell^2 &= m^2 + (18 - n)^2 \\ &\quad - m(18 - n).\end{aligned}$$

Write $u_j = 18 - jd$ for $j = 1, 2, 3$. Substituting the permutations

123, 132, 213, 231, 312, 321 of (u_1, u_2, u_3) for (m, n, ℓ) gives the nonzero candidates $d = 18, 2, 9, 5, 6, 6$, respectively. Positivity of u_3 requires $d < 6$, leaving only $d = 2$ and $d = 5$. The case $d = 0$ also gives a valid rhombus.

Thus the side sets are $\{18, 16, 14, 12\}$, $\{18, 13, 8, 3\}$, and $\{18, 18, 18, 18\}$.

The possible values of a non- AB side length are $\{3, 8, 12, 13, 14, 16, 18\}$, whose sum is 84.

Thus, the correct answer is **E**.

25. Let $m \geq 5$ be an odd integer, and let $D(m)$ denote the number of quadruples (a_1, a_2, a_3, a_4) of distinct integers with $1 \leq a_i \leq m$ for all i such that m divides $a_1 + a_2 + a_3 + a_4$. There is a polynomial

$$q(x) = c_3x^3 + c_2x^2 + c_1x + c_0$$

such that $D(m) = q(m)$ for all odd integers $m \geq 5$. What is c_1 ?

A -6

B -1

C 4

D 6

E 11

Solution:

Regard $1, 2, \dots, m$ as all residues modulo m . Without the distinctness condition, the first three entries are arbitrary and the fourth is determined, giving m^3 ordered quadruples.

Inclusion-exclusion over equal-coordinate partitions gives the remaining terms. There are 6 choices of one equal pair, each leaving m^2 solutions, for $-6m^2$. The 3 two-pair partitions contribute $+3m$. The 4 triple-and-single partitions have inclusion-exclusion coefficient 2 and contribute $+8m$. Finally, the all-equal partition has coefficient -6 and one solution. (Here odd m makes 2 and 4 invertible modulo m .) Therefore

$$\begin{aligned} D(m) &= m^3 - 6m^2 \\ &\quad + 11m - 6 \\ &= (m-1)(m-2) \\ &\quad \cdot (m-3). \end{aligned}$$

Hence $c_1 = 11$.

Thus, the correct answer is **E**.

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1. What is the value of

$$1234 + 2341 + 3412 + 4123?$$

A 10,000

B 10,010

C 10,110

D 11,000

E 11,110

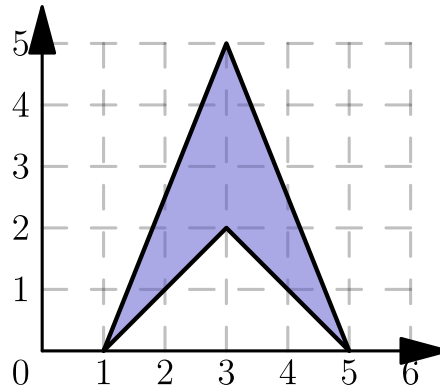
Solution:

Each of the digits 1, 2, 3, 4 appears exactly once in the thousands, hundreds, tens, and units columns. So each column sums to $1 + 2 + 3 + 4 = 10$.

The total is therefore $10 \cdot 1111 = 11110$.

Thus, the correct answer is **E**.

2. What is the area of the shaded figure shown below?



A 4

B 6

C 8

D 10

E 12

Solution:

The outer triangle has vertices $(1, 0)$, $(3, 5)$, and $(5, 0)$, giving base 4 and height 5, so its area is $\frac{1}{2} \cdot 4 \cdot 5 = 10$.

Removed from it is the triangle with vertices $(1, 0)$, $(3, 2)$, and $(5, 0)$, which has base 4 and height 2, so area $\frac{1}{2} \cdot 4 \cdot 2 = 4$.

The shaded area is $10 - 4 = 6$.

Thus, the correct answer is **B**.

3. At noon on a certain day, Minneapolis is N degrees warmer than St. Louis. At 4:00 the temperature in Minneapolis has fallen by 5 degrees while the temperature in St. Louis has risen by 3 degrees, at which time the temperatures in the two cities differ by 2 degrees. What is the product of all possible values of N ?

A 10

B 30

C 60

D 100

E 120

Solution:

At noon the gap is N . Minneapolis then loses 5 degrees and St. Louis gains 3, so the gap changes by 8. The new absolute difference is $|N - 8| = 2$.

This gives $N = 10$ or $N = 6$, whose product is 60.

Thus, the correct answer is **C**.

4. Let $n = 8^{2022}$. Which of the following is equal to

$$\frac{n}{4}?$$

A 4^{1010}

B 2^{2022}

C 8^{2018}

D 4^{3031}

E 4^{3032}

Solution:

Write $n = 8^{2022} = 2^{6066}$. Then

$$\frac{n}{4} = \frac{2^{6066}}{2^2} = 2^{6064}.$$

Since $2^{6064} = 4^{3032}$, this matches the last option.

Thus, the correct answer is **E**.

5. Call a fraction $\frac{a}{b}$, not necessarily in the simplest form, *special* if a and b are positive integers whose sum is 15. How many distinct integers can be written as the sum of two, not necessarily different, special fractions?

A 9

B 10

C 11

D 12

E 13

Solution:

Listing $\frac{a}{15-a}$ for $1 \leq a \leq 14$, the integers are 2, 4, 14; the fractions with fractional part $\frac{1}{2}$ are $\frac{1}{2}, \frac{3}{2}, \frac{13}{2}$; and $\frac{1}{4}, \frac{11}{4}$ have complementary fractional parts $\frac{1}{4}, \frac{3}{4}$.

The remaining fractional parts are $\frac{1}{14}, \frac{2}{13}, \frac{4}{11}, \frac{2}{3}, \frac{7}{8}, \frac{1}{7}$, and none of their complements occurs. Thus only the integer, half-integer, and quarter cases can give integer sums.

Integer pairs give 4, 6, 8, 16, 18, 28. Half-integer pairs $(\frac{1}{2}, \frac{3}{2}, \frac{13}{2})$ give 1, 2, 3, 7, 8, 13.

The quarter pair $\frac{1}{4} + \frac{11}{4}$ gives 3.

The distinct integers are 1, 2, 3, 4, 6, 7, 8, 13, 16, 18, 28, a total of 11.

Thus, the correct answer is **C**.

6. The greatest prime number that is a divisor of 16,384 is 2 because $16,384 = 2^{14}$. What is the sum of the digits of the greatest prime number that is a divisor of 16,383?

- ☐ A 3
- ☐ B 7
- ☒ C 10
- ☐ D 16
- ☐ E 22

Solution:

Since $16,383 = 2^{14} - 1 = (2^7 - 1)(2^7 + 1) = 127 \cdot 129$. Then $129 = 3 \cdot 43$, so $16,383 = 3 \cdot 43 \cdot 127$.

The greatest prime factor is 127, whose digit sum is $1 + 2 + 7 = 10$.

Thus, the correct answer is **C**.

7. Which of the following conditions is sufficient to guarantee that integers x , y , and z satisfy the equation

$$x(x - y) + y(y - z) + z(z - x) = 1?$$

- A $x > y$ and $y = z$
- B $x = y - 1$ and $y = z - 1$
- C $x = z + 1$ and $y = x + 1$
- D $x = z$ and $y - 1 = x$**
- E $x + y + z = 1$

Solution:

The expression satisfies

$$\begin{aligned} & 2[x(x - y) + y(y - z) + z(z - x)] \\ &= (x - y)^2 + (y - z)^2 + (z - x)^2. \end{aligned}$$

So the equation holds exactly when this sum of squares equals 2.

Since the three differences sum to 0, this requires two of them to be ± 1 and one to be 0.

Option D gives $z - x = 0$, $x - y = -1$, and $y - z = 1$, so the squares are $1 + 1 + 0 = 2$. This works for all such integers.

Thus, the correct answer is **D**.

8. The product of the lengths of the two congruent sides of an obtuse isosceles triangle is equal to the product of the base and twice the triangle's height to the base. What is the measure, in degrees, of the vertex angle of this triangle?

A 105

B 120

C 135

D 150

E 165

Solution:

Let the congruent sides have length s , the base be b , and the height to the base be h . The given condition is $s^2 = 2bh$.

The area equals $\frac{1}{2}bh$ and also $\frac{1}{2}s^2 \sin \theta$, where θ is the vertex angle. So $bh = s^2 \sin \theta$.

Substituting $s^2 = 2bh$ gives $bh = 2bh \sin \theta$, so $\sin \theta = \frac{1}{2}$. Since the triangle is obtuse, $\theta = 150^\circ$.

Thus, the correct answer is **D**.

9. Triangle ABC is equilateral with side length 6. Suppose that O is the center of the inscribed circle of this triangle. What is the area of the circle passing through A , O , and C ?

A 9π

B 12π

C 18π

D 24π

E 27π

Solution:

For an equilateral triangle, O is also the circumcenter, so $OA = OC = \frac{6}{\sqrt{3}} = 2\sqrt{3}$.

The central angle $\angle AOC = 120^\circ$.

In triangle AOC , side $AC = 6$ is opposite the 120° angle, so the circumradius R' of this triangle satisfies

$$2R' = \frac{6}{\sin 120^\circ} = 4\sqrt{3},$$

giving $R' = 2\sqrt{3}$.

The area of the circle is $\pi(2\sqrt{3})^2 = 12\pi$.

Thus, the correct answer is **B**.

10. What is the sum of all possible values of t between 0 and 360 such that the triangle in the coordinate plane whose vertices are $(\cos 40^\circ, \sin 40^\circ)$, $(\cos 60^\circ, \sin 60^\circ)$, and $(\cos t^\circ, \sin t^\circ)$ is isosceles?

A 100

B 150

C 330

D 360

E 380

Solution:

The three points lie on the unit circle at angles 40° , 60° , and t° . A chord's length depends only on the angular separation of its endpoints.

If the third point is equidistant from the other two, it lies on the perpendicular bisector: $t = 50$ or $t = 230$.

If its distance to 40° equals the fixed chord (separation 20°), then $t = 20$ (since $t = 60$ is degenerate). If its distance to 60° matches, then $t = 80$ (since $t = 40$ is degenerate).

The valid values are 50, 230, 20, 80, summing to 380.

Thus, the correct answer is **E**.

11. Una rolls 6 standard 6-sided dice simultaneously and calculates the product of the 6 numbers obtained. What is the probability that the product is divisible by 4?

A $\frac{3}{4}$

B $\frac{57}{64}$

C $\frac{59}{64}$

D $\frac{187}{192}$

E $\frac{63}{64}$

Solution:

The product fails to be divisible by 4 when it has at most one factor of 2. Each die is odd with probability $\frac{1}{2}$, contributes exactly one factor of 2 (a 2 or 6) with probability $\frac{1}{3}$, and two factors (a 4) with probability $\frac{1}{6}$.

All six odd: $\left(\frac{1}{2}\right)^6 = \frac{1}{64}$. Exactly one die a 2 or 6 and the rest odd: $6 \cdot \frac{1}{3} \cdot \left(\frac{1}{2}\right)^5 = \frac{1}{16} = \frac{4}{64}$.

The complement is $\frac{1}{64} + \frac{4}{64} = \frac{5}{64}$, so the answer is $1 - \frac{5}{64} = \frac{59}{64}$.

Thus, the correct answer is **C**.

12. For n a positive integer, let $f(n)$ be the quotient obtained when the sum of all positive divisors of n is divided by n . For example,

$$\begin{aligned} f(14) &= (1 + 2 + 7 + 14) \div 14 \\ &= \frac{12}{7}. \end{aligned}$$

What is $f(768) - f(384)$?

A $\frac{1}{768}$

B $\frac{1}{192}$

C 1

D $\frac{4}{3}$

E $\frac{8}{3}$

Solution:

Since $768 = 2^8 \cdot 3$, its divisor sum is $(2^9 - 1)(1 + 3) = 511 \cdot 4 = 2044$, so
 $f(768) = \frac{2044}{768} = \frac{511}{192}$.

Since $384 = 2^7 \cdot 3$, its divisor sum is $(2^8 - 1)(1 + 3) = 255 \cdot 4 = 1020$, so
 $f(384) = \frac{1020}{384} = \frac{510}{192}$.

The difference is $\frac{511 - 510}{192} = \frac{1}{192}$.

Thus, the correct answer is **B**.

13. Let $c = \frac{2\pi}{11}$. What is the value of

$$\frac{\sin 3c \cdot \sin 6c \cdot \sin 9c \cdot \sin 12c \cdot \sin 15c}{\sin c \cdot \sin 2c \cdot \sin 3c \cdot \sin 4c \cdot \sin 5c}?$$

A -1

B $-\frac{\sqrt{11}}{5}$

C $\frac{\sqrt{11}}{5}$

D $\frac{10}{11}$

E 1

Solution:

Write each angle as $kc = \frac{2\pi k}{11}$. Reducing modulo 2π , $\sin 12c = \sin c$ and $\sin 15c = \sin 4c$.

So the numerator is $\sin 3c \cdot \sin 6c \cdot \sin 9c \cdot \sin c \cdot \sin 4c$. Cancelling the common factors $\sin c, \sin 3c, \sin 4c$ leaves $\frac{\sin 6c \cdot \sin 9c}{\sin 2c \cdot \sin 5c}$.

Now $\sin 9c = \sin(2\pi - 2c) = -\sin 2c$ and $\sin 6c = \sin(2\pi - 5c) = -\sin 5c$, so the ratio equals $\frac{(-\sin 5c)(-\sin 2c)}{\sin 2c \cdot \sin 5c} = 1$.

Thus, the correct answer is **E**.

14. Suppose that $P(z)$, $Q(z)$, and $R(z)$ are polynomials with real coefficients, having degrees 2, 3, and 6, respectively, and constant terms 1, 2, and 3, respectively. Let N be the number of distinct complex numbers z that satisfy the equation

$$P(z) \cdot Q(z) = R(z).$$

What is the minimum possible value of N ?

- ☐ A 0
- ☒ B 1
- ☐ C 2
- ☐ D 3
- ☐ E 5

Solution:

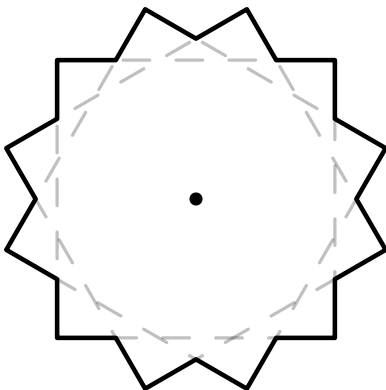
Let $D(z) = P(z)Q(z) - R(z)$. Since PQ has degree 5 and R has degree 6, the degree of D is 6. Its constant term is $1 \cdot 2 - 3 = -1 \neq 0$.

Because R is otherwise unconstrained, D can be made equal to any real degree-6 polynomial with constant term -1 , for instance $-(z - 1)^6$.

Such a polynomial has a single distinct root, so the minimum is $N = 1$.

Thus, the correct answer is **B**.

15. Three identical square sheets of paper each with side length 6 are stacked on top of each other. The middle sheet is rotated clockwise 30° about its center and the top sheet is rotated clockwise 60° about its center, resulting in the 24-sided polygon shown in the figure below. The area of this polygon can be expressed in the form $a - b\sqrt{c}$, where a , b , and c are positive integers, and c is not divisible by the square of any prime. What is $a + b + c$?



- A 75
- B 93
- C 96
- D 129
- E 147**

Solution:

Because the three squares are rotated by 0° , 30° , and 60° , the figure has 12-fold symmetry. Its 24 vertices alternate every 15° : outer vertices are the square corners at distance $3\sqrt{2}$ from the center, and inner vertices are edge crossings at distance $2\sqrt{3}$.

Connecting the center to all 24 vertices splits the polygon into 24 triangles, each with sides $3\sqrt{2}$ and $2\sqrt{3}$ and included angle 15° . The total area is

$$\begin{aligned}
 & 24 \cdot \frac{1}{2} (3\sqrt{2})(2\sqrt{3}) \sin 15^\circ \\
 &= 72\sqrt{6} \cdot \frac{\sqrt{6} - \sqrt{2}}{4}.
 \end{aligned}$$

This simplifies to $18\sqrt{6}(\sqrt{6} - \sqrt{2}) = 108 - 36\sqrt{3}$, so $a + b + c = 108 + 36 + 3 = 147$.

Thus, the correct answer is **E**.

16. Suppose a, b, c are positive integers such that

$$a + b + c = 23$$

and

$$\begin{aligned} \gcd(a, b) + \gcd(b, c) \\ + \gcd(c, a) = 9. \end{aligned}$$

What is the sum of all possible distinct values of $a^2 + b^2 + c^2$?

A 259

B 438

C 516

D 625

E 687

Solution:

The condition is symmetric, so assume $a \leq b \leq c$. Then $a \leq 7$, and for each fixed a we have $a \leq b \leq \lfloor \frac{23-a}{2} \rfloor$ and $c = 23 - a - b$. The gcd sum can be evaluated as

$$\begin{aligned} \gcd(a, b) + \gcd(b, 23 - a) \\ + \gcd(a, 23 - b). \end{aligned}$$

Checking these short ranges, the possible gcd sums for $a = 1, 2, \dots, 7$ are, respectively, $\{3, 4, 13\}$, $\{4, 6, 10\}$, $\{3, 5, 6, 8, 9, 12\}$, $\{4, 6\}$, $\{3, 7, 8, 11\}$, $\{4, 6, 8\}$, and $\{9, 10\}$. The value 9 occurs only at $(a, b, c) = (3, 5, 15)$ and $(7, 7, 9)$.

The triple $(7, 7, 9)$ has gcd sum $7 + 1 + 1 = 9$ and $a^2 + b^2 + c^2 = 49 + 49 + 81 = 179$. The triple $(3, 5, 15)$ has gcd sum $1 + 5 + 3 = 9$ and $a^2 + b^2 + c^2 = 9 + 25 + 225 = 259$.

The sum of the distinct values is $179 + 259 = 438$.

Thus, the correct answer is **B**.

17. A bug starts at a vertex of a grid made of equilateral triangles of side length 1. At each step the bug moves in one of the 6 possible directions along the grid lines randomly and independently with equal probability. What is the probability that after 5 moves the bug never will have been more than 1 unit away from the starting position?

A $\frac{13}{108}$

B $\frac{7}{54}$

C $\frac{29}{216}$

D $\frac{4}{27}$

E $\frac{1}{16}$

Solution:

Staying within distance 1 means the bug is always at the origin or one of its 6 neighbors. From the origin, all 6 moves are allowed. From a neighbor, only 3 moves keep it in range: back to the origin, or to either of the two adjacent neighbors.

Let a_k and b_k count valid k -step paths ending at the origin and at a neighbor. Then $a_{k+1} = b_k$ and $b_{k+1} = 6a_k + 2b_k$, starting from $a_0 = 1, b_0 = 0$.

Iterating gives $b_1 = 6$, then $(a_2, b_2) = (6, 12), (a_3, b_3) = (12, 60), (a_4, b_4) = (60, 192), (a_5, b_5) = (192, 744)$. The total number of valid paths is $192 + 744 = 936$.

The probability is $\frac{936}{6^5} = \frac{936}{7776} = \frac{13}{108}$.

Thus, the correct answer is **A**.

18. Set $u_0 = \frac{1}{4}$, and for $k \geq 0$ let u_{k+1} be determined by the recurrence

$$u_{k+1} = 2u_k - 2u_k^2.$$

This sequence tends to a limit; call it L . What is the least value of k such that

$$|u_k - L| \leq \frac{1}{2^{1000}}?$$

A 10

B 87

C 123

D 329

E 401

Solution:

The limit satisfies $L = 2L - 2L^2$, giving $L = \frac{1}{2}$. Let $v_k = 1 - 2u_k$. Then

$$\begin{aligned} v_{k+1} &= 1 - 2u_{k+1} \\ &= 1 - 4u_k + 4u_k^2 \\ &= (1 - 2u_k)^2 = v_k^2. \end{aligned}$$

Since $v_0 = 1 - 2 \cdot \frac{1}{4} = \frac{1}{2}$, we get $v_k = \left(\frac{1}{2}\right)^{2^k}$, so $|u_k - L| = \frac{|v_k|}{2} = 2^{-2^k-1}$.

We need $2^k + 1 \geq 1000$, i.e. $2^k \geq 999$. The least such k is 10, since $2^{10} = 1024$.

Thus, the correct answer is **A**.

19. Regular polygons with 5, 6, 7, and 8 sides are inscribed in the same circle. No two of the polygons share a vertex, and no three of their sides intersect at a common point. At how many points inside the circle do two of their sides intersect?

- A 52
- B 56
- C 60
- D 64
- E 68**

Solution:

For two convex polygons inscribed in the same circle with no shared vertices, each side of the smaller polygon crosses the larger polygon's boundary exactly twice, so they meet at $2 \min(m, n)$ points.

Summing over all pairs: (5, 6), (5, 7), (5, 8) give 10 each; (6, 7), (6, 8) give 12 each; (7, 8) gives 14.

The total is $3 \cdot 10 + 2 \cdot 12 + 14 = 68$.

Thus, the correct answer is **E**.

20. A cube is constructed from 4 white unit cubes and 4 blue unit cubes. How many different ways are there to construct the $2 \times 2 \times 2$ cube using these smaller cubes? (Two constructions are considered the same if one can be rotated to match the other.)

- ☒ A 7
- ☐ B 8
- ☐ C 9
- ☐ D 10
- ☐ E 11

Solution:

By Burnside's lemma, the count is the average number of 4-blue colorings fixed by each of the 24 rotations acting on the 8 cubies.

The identity fixes $\binom{8}{4} = 70$. The 6 face quarter-turns fix 2 each (12). The 3 face half-turns fix 6 each (18). The 8 vertex rotations fix 4 each (32). The 6 edge half-turns fix 6 each (36).

The total is $70 + 12 + 18 + 32 + 36 = 168$, and $\frac{168}{24} = 7$.

Thus, the correct answer is **A**.

21. For real numbers x , let

$$\begin{aligned} P(x) = & 1 + \cos(x) + i \sin(x) \\ & - \cos(2x) - i \sin(2x) \\ & + \cos(3x) + i \sin(3x) \end{aligned}$$

where $i = \sqrt{-1}$. For how many values of x with $0 \leq x < 2\pi$ does

$$P(x) = 0?$$

☒ A 0

☐ B 1

☐ C 2

☐ D 3

☐ E 4

Solution:

Group by Euler's formula: $P(x) = 1 + e^{ix} - e^{2ix} + e^{3ix}$. The imaginary part is

$$\begin{aligned} & \sin x - \sin 2x + \sin 3x \\ &= (\sin x + \sin 3x) - \sin 2x \\ &= \sin 2x(2 \cos x - 1). \end{aligned}$$

This vanishes when $\sin 2x = 0$ (so $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$) or $\cos x = \frac{1}{2}$ (so $x = \frac{\pi}{3}, \frac{5\pi}{3}$).

Checking the real part $1 + \cos x - \cos 2x + \cos 3x$ at each of these values gives ± 2 or 1, never 0. So no x makes $P(x) = 0$.

Thus, the correct answer is **A**.

22. Right triangle ABC has side lengths $BC = 6$, $AC = 8$, and $AB = 10$. A circle centered at O is tangent to line BC at B and passes through A . A circle centered at P is tangent to line AC at A and passes through B . What is OP ?

A $\frac{23}{8}$

B $\frac{29}{10}$

C $\frac{35}{12}$

D $\frac{73}{25}$

E 3

Solution:

Place $C = (0, 0)$, $B = (6, 0)$, and $A = (0, 8)$, so the right angle is at C .

Circle O is tangent to line BC (the x -axis) at B , so $O = (6, k)$. Setting $OA = OB$ gives $36 + (k - 8)^2 = k^2$, so $k = \frac{25}{4}$ and $O = (6, \frac{25}{4})$.

Circle P is tangent to line AC (the y -axis) at A , so $P = (h, 8)$. Setting $PB = PA$ gives $(h - 6)^2 + 64 = h^2$, so $h = \frac{25}{3}$ and $P = (\frac{25}{3}, 8)$.

$$\text{Then } OP = \sqrt{\left(\frac{7}{3}\right)^2 + \left(\frac{7}{4}\right)^2} = 7\sqrt{\frac{25}{144}} = \frac{35}{12}.$$

Thus, the correct answer is **C**.

23. What is the average number of pairs of consecutive integers in a randomly selected subset of 5 distinct integers chosen from the set $\{1, 2, 3, \dots, 30\}$? (For example the set $\{1, 17, 18, 19, 30\}$ has 2 pairs of consecutive integers.)

A $\frac{2}{3}$

B $\frac{29}{36}$

C $\frac{5}{6}$

D $\frac{29}{30}$

E 1

Solution:

For each of the 29 adjacent pairs $(i, i + 1)$, let an indicator be 1 if both are in the subset. The probability of this is $\frac{5}{30} \cdot \frac{4}{29} = \frac{2}{87}$.

By linearity of expectation, the expected number of consecutive pairs is $29 \cdot \frac{2}{87} = \frac{2}{3}$.

Thus, the correct answer is **A**.

24. Triangle ABC has side lengths $AB = 11$, $BC = 24$, and $CA = 20$. The bisector of $\angle BAC$ intersects $\overline{BC}$ in point D , and intersects the circumcircle of $\triangle ABC$ in point $E \neq A$. The circumcircle of $\triangle BED$ intersects the line AB in points B and $F \neq B$. What is CF ?

A 28

B $20\sqrt{2}$

C 30

D 32

E $20\sqrt{3}$

Solution:

Points A, D, E are collinear on the bisector, and A, B, F are collinear on line AB . The power of A with respect to the circle through B, E, D gives $AB \cdot AF = AD \cdot AE$.

Since $\angle BAE = \angle DAC$ and $\angle AEB = \angle ACB$ (subtending AB), triangles ABE and ADC are similar, so $AD \cdot AE = AB \cdot AC$. Therefore $AF = AC = 20$.

Place $A = (0, 0)$, $B = (11, 0)$. From $CA = 20$, $CB = 24$, point $C = \left(-\frac{5}{2}, \frac{\sqrt{1575}}{2}\right)$.

Point F lies on ray AB with $AF = 20$, so $F = (20, 0)$.

Then $CF^2 = \left(20 + \frac{5}{2}\right)^2 + \frac{1575}{4} = \frac{2025}{4} + \frac{1575}{4} = 900$, so $CF = 30$.

Thus, the correct answer is **C**.

25. For n a positive integer, let $R(n)$ be the sum of the remainders when n is divided by 2, 3, 4, 5, 6, 7, 8, 9, and 10. For example,

$$R(15) = 1 + 0 + 3 + 0 + 3 + 1 + 7 + 6 + 5 = 26.$$

How many two-digit positive integers n satisfy

$$R(n) = R(n + 1)?$$

- ☐ A 0
- ☐ B 1
- ☒ C 2
- ☐ D 3
- ☐ E 4

Solution:

Going from n to $n + 1$, each remainder $n \bmod m$ increases by 1 unless $m \mid n + 1$, in which case it drops from $m - 1$ to 0. So

$$\begin{aligned} R(n + 1) - R(n) &= 9 - \sum_{\substack{2 \leq m \leq 10 \\ m \mid n+1}} m. \end{aligned}$$

Distinct integers from 2 through 10 can sum to 9 only as $\{9\}$, $\{2, 7\}$, $\{3, 6\}$, $\{4, 5\}$, or $\{2, 3, 4\}$. Divisibility closure eliminates every case except $\{2, 7\}$: for example, a multiple of 9 also has divisor 3, and a multiple of 6 also has divisors 2 and 3. Thus $n + 1$ must be a multiple of 14 with no other divisor in $\{2, \dots, 10\}$.

Testing the multiples 14, 28, 42, 56, 70, 84, 98 in the required range leaves only 14 and 98. Hence $n = 13$ or $n = 97$, giving 2 values.

Thus, the correct answer is **C**.

Problems: <https://live.poshenloh.com/past-contests/amc12/2021D>

