AMC 12 SOLUTION ARCHIVE

2020

12A and 12B full explanations

Contents

AMC 12A full solutions	26 pages	**starts p. 2**
AMC 12B full solutions	27 pages	**starts p. 28**

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2020 AMC 12A Solutions

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1. Carlos took 70% of a whole pie. Maria took one third of the remainder. What portion of the whole pie was left?

A 10%

B 15%

C 20%

D 30%

E 35%

Solution:

After Carlos takes 70%, the remaining portion is 30%.

Maria takes one third of this, namely $\frac{1}{3} \cdot 30\% = 10\%$, leaving $30\% - 10\% = 20\%$.

Thus, **C** is the correct answer.

2. The acronym AMC is shown in the rectangular grid below with grid lines spaced 1 unit apart. In units, what is the sum of the lengths of the line segments that form the acronym AMC?

A 17

B $15 + 2\sqrt{2}$

C $13 + 4\sqrt{2}$

D $11 + 6\sqrt{2}$

E 21

Solution:

Split each letter into its segments. The A is a diagonal of length $2\sqrt{2}$, a vertical of length 2, and a crossbar of length 1.

The M has two verticals of length 2 and two diagonals of length $\sqrt{2}$ each. The C is three sides of length 2.

The straight pieces total $2 + 1 + 2 + 2 + 2 + 2 + 2 = 13$, and the diagonal pieces total $2\sqrt{2} + \sqrt{2} + \sqrt{2} = 4\sqrt{2}$.

The sum is $13 + 4\sqrt{2}$.

Thus, **C** is the correct answer.

3. A driver travels for 2 hours at 60 miles per hour, during which her car gets 30 miles per gallon of gasoline. She is paid $\$0.50$ per mile, and her only expense is gasoline at $\$2.00$ per gallon. What is her net rate of pay, in dollars per hour, after this expense?

A 20

B 22

C 24

D 25

E 26

Solution:

In 2 hours she drives 120 miles, earning $120 \cdot \$0.50 = \$60.$

She uses $120 \div 30 = 4$ gallons, costing $4 \cdot \$2.00 = \$8.$

Her net earnings are $\$60 - \$8 = \$52,$ so her rate is $\$52 \div 2 = \$26$ per hour.

Thus, **E** is the correct answer.

4. How many 4-digit positive integers (that is, integers between 1000 and 9999, inclusive) having only even digits are divisible by 5?

A 80

B 100

C 125

D 200

E 500

Solution:

To be divisible by 5 the last digit is 0 or 5, and to be even it must be 0. So the units digit is fixed.

The leading digit is a nonzero even digit: $2, 4, 6, 8$ give 4 choices. Each of the two middle digits is any even digit $0, 2, 4, 6, 8$, giving 5 choices each.

The total is $4 \cdot 5 \cdot 5 \cdot 1 = 100$.

Thus, **B** is the correct answer.

5. The 25 integers from -10 to 14, inclusive, can be arranged to form a 5-by-5 square in which the sum of the numbers in each row, the sum of the numbers in each column, and the sum of the numbers along each of the main diagonals are all the same. What is the value of this common sum?

A 2

B 5

C 10

D 25

E 50

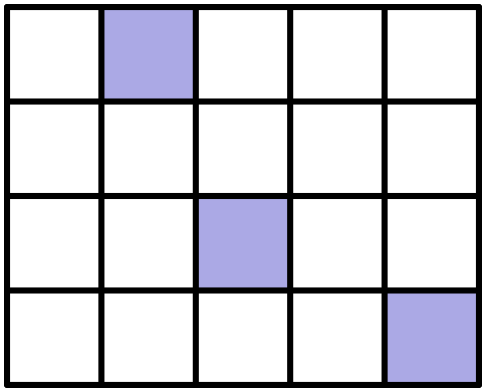
Solution:

The sum of the 25 integers is $\dfrac{(-10+14)\cdot 25}{2} = 50.$

The five rows each have the same sum and together account for the total, so each row sums to $50 \div 5 = 10.$

Thus, **C** is the correct answer.

6. In the plane figure shown below, 3 of the unit squares have been shaded. What is the least number of additional unit squares that must be shaded so that the resulting figure has two lines of symmetry?



A 4

B 5

C 6

D 7

E 8

Solution:

For both symmetries, the lines must be the vertical and horizontal center lines of the 5-by-4 grid. Every shaded square then forces the squares obtained by reflecting it across each line.

The top square lies off-center, so its reflection group has 4 squares, requiring 3 more. The middle square sits on the central column, so its group has 2 squares, requiring 1 more. The bottom-right square again has a group of 4, requiring 3 more.

The least number of additional squares is $3 + 1 + 3 = 7.$

Thus, **D** is the correct answer.

7. Seven cubes, whose volumes are $1, 8, 27, 64, 125, 216,$ and 343 cubic units, are stacked vertically to form a tower in which the volumes of the cubes decrease from bottom to top. Except for the bottom cube, the bottom face of each cube lies completely on top of the cube below it. What is the total surface area of the tower (including the bottom) in square units?

A 644

B 658

C 664

D 720

E 749

Solution:

The side lengths are $1, 2, \ldots, 7$. The four side faces of cube k contribute $4k^2$, so the vertical faces total $4(1^2 + 2^2 + \cdots + 7^2) = 4 \cdot 140 = 560$.

Viewed from directly above, every upward-facing horizontal patch projects onto the 7×7 base without overlap, giving 49. Viewed from below, the same is true, giving another 49.

The total surface area is $560 + 49 + 49 = 658$.

Thus, **B** is the correct answer.

8. What is the median of the following list of 4040 numbers?

$$1, 2, 3, \ldots, 2020, 1^2, 2^2, 3^2, \ldots, 2020^2$$

A 1974.5

B 1975.5

C 1976.5

D 1977.5

E 1978.5

Solution:

The median is the average of the 2020th and 2021st smallest values.

The perfect squares that are at most 2020 are $1^2, \ldots, 44^2$ (since $44^2 = 1936$ and $45^2 = 2025$), so there are 44 of them.

Among the list, the numbers ≤ 1976 are the 1976 integers $1, \ldots, 1976$ together with those 44 squares, totaling $1976 + 44 = 2020$.

Thus the 2020th value is 1976 and the 2021st value is 1977, making the median $\dfrac{1976 + 1977}{2} = 1976.5$.

Thus, **C** is the correct answer.

9. How many solutions does the equation $\tan(2x) = \cos\left(\frac{x}{2}\right)$ have on the interval $[0, 2\pi]$?

A 1

B 2

C 3

D 4

E 5

Solution:

On $[0, 2\pi]$, the graph of $\cos\left(\frac{x}{2}\right)$ is a single arc decreasing from 1 down to -1.

The function $\tan(2x)$ has period $\frac{\pi}{2}$ with vertical asymptotes at $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$. These split the interval into five branches. On every branch $\tan(2x)$ is strictly increasing, while $\cos(\frac{x}{2})$ is decreasing, so there is at most one intersection per branch.

Each of the three interior branches runs from $-\infty$ to $+\infty$, so each has one intersection. On the first branch, $\tan(0) = 0 < 1 = \cos(0)$ and the tangent tends to $+\infty$. On the last, the tangent starts at $-\infty$ and ends at $0 > -1 = \cos\pi$. Thus the two outer branches also have one intersection each, for 5 total.

Thus, **E** is the correct answer.

10. There is a unique positive integer n such that

$$\log_2(\log_{16} n) = \log_4(\log_4 n).$$

What is the sum of the digits of n?

A 4

B 7

C 8

D 11

E 13

Solution:

Since $\log_{16} n = \frac{1}{2}\log_4 n$, set $y = \log_4 n$. The equation becomes $\log_2\left(\frac{y}{2}\right) = \log_4 y = \frac{1}{2}\log_2 y$.

Multiplying by 2 gives $\log_2\left(\frac{y}{2}\right)^2 = \log_2 y$, so $\left(\frac{y}{2}\right)^2 = y$, which yields $y = 4$.

Then $\log_4 n = 4$, so $n = 4^4 = 256$, and the digit sum is $2 + 5 + 6 = 13$.

Thus, **E** is the correct answer.

11. A frog sitting at the point $(1, 2)$ begins a sequence of jumps, where each jump is parallel to one of the coordinate axes and has length 1, and the direction of each jump (up, down, right, or left) is chosen independently at random. The sequence ends when the frog reaches a side of the square with vertices $(0, 0)$, $(0, 4)$, $(4, 4)$, and $(4, 0)$. What is the probability that the sequence of jumps ends on a vertical side of the square?

A $\frac{1}{2}$

B $\frac{5}{8}$

C $\frac{2}{3}$

D $\frac{3}{4}$

E $\frac{7}{8}$

Solution:

Let $P(x, y)$ be the probability of ending on a vertical side. On a vertical side $P = 1$, on a horizontal side $P = 0$, and at an interior point P is the average of its four neighbors.

By left-right symmetry $P(2, 2) = \frac{1}{2}$. Let $a = P(1, 2), b = P(1, 1) = P(1, 3)$, and $c = P(2, 1) = P(2, 3)$. Then

$a = \frac{1}{4}\left(1 + \frac{1}{2} + 2b\right)$, $b = \frac{1}{4}(1 + c + a)$, and $c = \frac{1}{4}\left(2b + \frac{1}{2}\right)$.

Substituting gives $b = \frac{1}{2}$, hence $a = \frac{3}{8} + \frac{1}{2}b = \frac{5}{8}$.

Thus, **B** is the correct answer.

12. Line ℓ in the coordinate plane has the equation $3x - 5y + 40 = 0$. This line is rotated $45°$ counterclockwise about the point $(20, 20)$ to obtain line k. What is the x-coordinate of the x-intercept of line k?

A 10

B 15

C 20

D 25

E 30

Solution:

Note $(20, 20)$ satisfies $3x - 5y + 40 = 0$, so it is on ℓ and remains on k. The slope of ℓ is $\frac{3}{5}$.

Rotating by $45°$ gives slope $\dfrac{\frac{3}{5} + 1}{1 - \frac{3}{5}} = \dfrac{\frac{8}{5}}{\frac{2}{5}} = 4.$

Line k is $y - 20 = 4(x - 20)$. Setting $y = 0$ gives $-20 = 4(x - 20)$, so $x - 20 = -5$ and $x = 15$.

Thus, **B** is the correct answer.

13. There are integers a, b, and c, each greater than 1, such that

$$\sqrt[a]{N\sqrt[b]{N\sqrt[c]{N}}} = \sqrt[36]{N^{25}}$$

for all $N > 1$. What is b?

A 2

B 3

C 4

D 5

E 6

Solution:

The left side equals N raised to the exponent $\frac{1}{a} + \frac{1}{ab} + \frac{1}{abc}$, which must equal $\frac{25}{36}$.

Because $b, c \geq 2$, this exponent is at most $\frac{7}{4a}$. If $a \geq 3$, it is at most $\frac{7}{12} < \frac{25}{36}$, so $a = 2$. The equation becomes $\frac{1}{2b} + \frac{1}{2bc} = \frac{7}{36}$.

Clearing denominators gives $c(7b - 18) = 18$. Set $d = 7b - 18$, so d is a positive divisor of 18. Among $1, 2, 3, 6, 9, 18$, only $d = 3$ makes $b = \frac{d + 18}{7}$ an integer. Hence $b = 3$ and $c = 6$.

Thus, **B** is the correct answer.

14. Regular octagon $ABCDEFGH$ has area n. Let m be the area of quadrilateral $ACEG$. What is $\frac{m}{n}$?

A $\frac{\sqrt{2}}{4}$

B $\frac{\sqrt{2}}{2}$

C $\frac{3}{4}$

D $\frac{3\sqrt{2}}{5}$

E $\frac{2\sqrt{2}}{3}$

Solution:

The four vertices A, C, E, G form a square, since they are every other vertex of the regular octagon.

Taking a unit circumradius, the octagon's area is $2\sqrt{2}$ and the square $ACEG$ has diagonal equal to the circle's diameter, giving area 2.

The ratio is $\frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$.

Thus, **B** is the correct answer.

15. In the complex plane, let A be the set of solutions to $z^3 - 8 = 0$ and let B be the set of solutions to $z^3 - 8z^2 - 8z + 64 = 0$. What is the greatest distance between a point of A and a point of B?

A $2\sqrt{3}$

B 6

C 9

D $2\sqrt{21}$

E $9 + \sqrt{3}$

Solution:

The set A consists of the cube roots of $8: 2, -1 + i\sqrt{3}$, and $-1 - i\sqrt{3}$.

Factoring by grouping, $z^3 - 8z^2 - 8z + 64 = z^2(z - 8) - 8(z - 8) = (z - 8)(z^2 - 8)$, so $B = \{8, 2\sqrt{2}, -2\sqrt{2}\}$, all real.

The greatest distance is from $-1 \pm i\sqrt{3}$ to $8: \sqrt{(8 - (-1))^2 + (\sqrt{3})^2} = \sqrt{81 + 3} = \sqrt{84} = 2\sqrt{21}$.

Thus, **D** is the correct answer.

16. A point is chosen at random within the square in the coordinate plane whose vertices are $(0, 0), (2020, 0), (2020, 2020)$, and $(0, 2020)$. The probability that the point is within d units of a lattice point is $\frac{1}{2}$. (A point (x, y) is a lattice point if x and y are both integers.) What is d to the nearest tenth?

A 0.3

B 0.4

C 0.5

D 0.6

E 0.7

Solution:

By periodicity it suffices to consider one unit cell with a lattice point at each corner. The region within d of a corner consists of four quarter-disks of radius d, forming one full disk of area πd^2. The resulting value will be less than $\frac{1}{2}$, so these quarter-disks do not overlap.

Setting $\pi d^2 = \frac{1}{2}$ gives $d = \sqrt{\dfrac{1}{2\pi}} \approx 0.399.$

To the nearest tenth, $d = 0.4.$

Thus, **B** is the correct answer.

17. The vertices of a quadrilateral lie on the graph of $y = \ln x$, and the x-coordinates of these vertices are consecutive positive integers. The area of the quadrilateral is $\ln \frac{91}{90}$. What is the x-coordinate of the leftmost vertex?

A 6

B 7

C 10

D 12

E 13

Solution:

Let the vertices have x-coordinates $n, n+1, n+2, n+3$ with y-coordinates $\ln$ of those values. Applying the shoelace formula and simplifying, the area is $\ln \frac{(n+1)(n+2)}{n(n+3)}$.

Setting $\frac{(n+1)(n+2)}{n(n+3)} = \frac{91}{90}$ gives $1 + \frac{2}{n^2+3n} = \frac{91}{90}$, so $n^2 + 3n = 180$.

Then $(n-12)(n+15) = 0$, so $n = 12$.

Thus, **D** is the correct answer.

18. Quadrilateral $ABCD$ satisfies $\angle ABC = \angle ACD = 90^\circ$, $AC = 20$, and $CD = 30$. Diagonals AC and BD intersect at point E, and $AE = 5$. What is the area of quadrilateral $ABCD$?

A 330

B 340

C 350

D 360

E 370

Solution:

Place $A = (0, 0)$ and $C = (20, 0)$. Since $\angle ACD = 90^\circ$, $D = (20, 30)$, and $E = (5, 0)$ because $AE = 5$.

Since $\angle ABC = 90^\circ$, B lies on the circle of radius 10 centered at $(10, 0)$. Line DE is $(5 + t,\ 2t)$; substituting gives $t^2 - 2t - 15 = 0$, so $t = 5$ or $t = -3$.

For E to lie between B and D, take $t = -3$, giving $B = (2, -6)$, a distance 6 below line AC.

Then $[ACD] = \frac{1}{2} \cdot 20 \cdot 30 = 300$ and $[ABC] = \frac{1}{2} \cdot 20 \cdot 6 = 60$, so the total area is 360.

Thus, **D** is the correct answer.

19. There exists a unique strictly increasing sequence of nonnegative integers $a_1 < a_2 < \cdots < a_k$ such that

$$\frac{2^{289}+1}{2^{17}+1} = 2^{a_1} + 2^{a_2} + \cdots + 2^{a_k}.$$

What is k?

A 117

B 136

C 137

D 273

E 306

Solution:

Let $x = 2^{17}$. Then $\dfrac{2^{289}+1}{2^{17}+1} = \dfrac{x^{17}+1}{x+1} = x^{16} - x^{15} + \cdots - x + 1$, an alternating sum of the 17 powers $x^0, x^1, \ldots, x^{16}$.

Pair each subtracted power with the added power just above it: $x^{m+1} - x^m = 2^{17m}(2^{17}-1) = 2^{17m} + 2^{17m+1} + \cdots + 2^{17m+16}$, a block of 17 consecutive powers of 2.

There are 8 such pairs, together with the leftover $+2^0$. The blocks occupy disjoint ranges, so the total number of powers is $8 \cdot 17 + 1 = 137$.

Thus, **C** is the correct answer.

20. Let T be the triangle in the coordinate plane with vertices $(0, 0)$, $(4, 0)$, and $(0, 3)$. Consider the following five isometries (rigid transformations) of the plane: rotations of 90°, 180°, and 270° counterclockwise around the origin, reflection across the x-axis, and reflection across the y-axis. How many of the 125 sequences of three of these transformations (not necessarily distinct) will return T to its original position? (For example, a 180° rotation, followed by a reflection across the x-axis, followed by a reflection across the y-axis will return T to its original position, but a 90° rotation, followed by a reflection across the x-axis, followed by another reflection across the x-axis will not return T to its original position.)

A 12

B 15

C 17

D 20

E 25

Solution:

Because T is a scalene right triangle, the only isometry carrying T to itself is the identity, so a sequence works exactly when the three transformations compose to the identity.

Let r be the 90° rotation and s reflection across the x-axis. The five allowed maps are r, r^2, r^3, s, r^2s; the missing nonidentity maps are the diagonal reflections rs, r^3s. In an ordered triple, the third map is forced by the first two and is allowed precisely when their product is one of the five.

Exactly 5 ordered pairs have product identity: each first map is paired with its inverse. A diagonal reflection requires one rotation and one axis reflection; r or r^3 may be paired on either side with s or r^2s, giving $2 \cdot 2 \cdot 2 = 8$ pairs. The remaining $25 - 5 - 8 = 12$ ordered pairs give valid sequences.

Thus, **A** is the correct answer.

21. How many positive integers n are there such that n is a multiple of 5, and the least common multiple of $5!$ and n equals 5 times the greatest common divisor of $10!$ and n?

A 12

B 24

C 36

D 48

E 72

Solution:

Write $n = 2^a 3^b 5^c 7^d \cdots$. Since $5! = 2^3 \cdot 3 \cdot 5$ has no other primes, n can only involve $2, 3, 5, 7$. Matching exponents in $\text{lcm}(5!, n) = 5 \cdot \gcd(10!, n)$:

For $2 : \max(3, a) = \min(8, a)$, so $3 \leq a \leq 8$ gives 6 values. For $3 : \max(1, b) = \min(4, b)$, so $1 \leq b \leq 4$ gives 4 values.

For $5 : \max(1, c) = 1 + \min(2, c)$ with $c \geq 1$, which forces $c = 3$, giving 1 value. For $7 : \max(0, d) = \min(1, d)$, so $d = 0$ or 1, giving 2 values.

The total is $6 \cdot 4 \cdot 1 \cdot 2 = 48$.

Thus, **D** is the correct answer.

22. Let (a_n) and (b_n) be the sequences of real numbers such that

$$(2+i)^n = a_n + b_n i$$

for all integers $n \geq 0$, where $i = \sqrt{-1}$. What is

$$\sum_{n=0}^{\infty} \frac{a_n b_n}{7^n}?$$

A $\frac{3}{8}$

B $\frac{7}{16}$

C $\frac{1}{2}$

D $\frac{9}{16}$

E $\frac{4}{7}$

Solution:

Since $(a_n + b_n i)^2 = a_n^2 - b_n^2 + 2a_n b_n i$, we have $a_n b_n = \frac{1}{2}\operatorname{Im}\left((2+i)^{2n}\right) = \frac{1}{2}\operatorname{Im}\left((3+4i)^n\right)$.

Therefore the sum is $\frac{1}{2}\operatorname{Im}\sum_{n=0}^{\infty}\left(\frac{3+4i}{7}\right)^n = \frac{1}{2}\operatorname{Im}\left(\frac{1}{1-\frac{3+4i}{7}}\right)$.

This equals $\frac{1}{2}\operatorname{Im}\left(\frac{7}{4-4i}\right) = \frac{1}{2}\operatorname{Im}\left(\frac{7(4+4i)}{32}\right) = \frac{1}{2}\cdot\frac{28}{32} = \frac{7}{16}$.

Thus, **B** is the correct answer.

23. Jason rolls three fair standard six-sided dice. Then he looks at the rolls and chooses a subset of the dice (possibly empty, possibly all three dice) to reroll. After rerolling, he wins if and only if the sum of the numbers face up on the three dice is exactly 7. Jason always plays to optimize his chances of winning. What is the probability that he chooses to reroll exactly two of the dice?

A $\frac{7}{36}$

B $\frac{5}{24}$

C $\frac{2}{9}$

D $\frac{17}{72}$

E $\frac{1}{4}$

Solution:

Rerolling one die, keeping two dice that sum to s, wins with probability $\frac{1}{6}$ when $s \leq 6$ and 0 otherwise. Rerolling two dice, keeping a die of value v, wins with probability equal to the number of ways two dice sum to $7-v$, over 36; this is largest when v is smallest. For $v = 1, 2, 3$, these probabilities are $\frac{5}{36}$, $\frac{4}{36}$, $\frac{3}{36}$, all greater than the reroll-all probability $\frac{15}{216} = \frac{5}{72}$; for $v \geq 4$, rerolling all is better.

Rerolling exactly two dice is strictly best precisely when the two smallest dice sum to at least 7 (so rerolling one cannot reach 7) while the smallest die is $1, 2$, or 3 (so keeping it beats rerolling all three).

Sort the roll as $u \leq v \leq w$. If $u = 1$, the only possibility is $(1, 6, 6)$, with 3 orderings. If $u = 2$, the possibilities $(2, 5, 5), (2, 5, 6), (2, 6, 6)$ have $3 + 6 + 3 = 12$ orderings. If $u = 3$, choose v, w with repetition from $\{4, 5, 6\}$; the six resulting triples have $3 + 6 + 6 + 3 + 6 + 3 = 27$ orderings. Thus there are $3 + 12 + 27 = 42$ qualifying ordered rolls out of 216, a probability of $\frac{42}{216} = \frac{7}{36}$.

Thus, **A** is the correct answer.

24. Suppose that $\triangle ABC$ is an equilateral triangle of side length s, with the property that there is a unique point P inside the triangle such that $AP = 1, BP = \sqrt{3}$, and $CP = 2$. What is s?

A $1 + \sqrt{2}$

B $\sqrt{7}$

C $\frac{8}{3}$

D $\sqrt{5 + \sqrt{5}}$

E $2\sqrt{2}$

Solution:

A point at distances p, q, r from the vertices of an equilateral triangle of side s satisfies $3(p^4 + q^4 + r^4 + s^4) = (p^2 + q^2 + r^2 + s^2)^2$.

With $p^2 = 1, q^2 = 3, r^2 = 4$, letting $S = s^2$ gives $3(26 + S^2) = (8 + S)^2$, so $S^2 - 8S + 7 = 0$ and $S = 1$ or $S = 7$.

A triangle of side 1 cannot contain a point at distance 2 from a vertex, so $S = 7$ and $s = \sqrt{7}$.

Thus, **B** is the correct answer.

25. The number $a = \dfrac{p}{q}$, where p and q are relatively prime positive integers, has the property that the sum of all real numbers x satisfying

$$\lfloor x \rfloor \cdot \{x\} = a \cdot x^2$$

is 420, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x and $\{x\} = x - \lfloor x \rfloor$ denotes the fractional part of x. What is $p + q$?

A 245

B 593

C 929

D 1331

E 1332

Solution:

There are no negative solutions, while $x = 0$ is always a solution. For $n \geq 1$ and $x \in [n, n+1)$, put $y = \frac{x}{n}$. The equation becomes $ay^2 - y + 1 = 0$. Its roots must be real, so $0 < a \leq \frac{1}{4}$.

If the two roots are $\alpha \leq \beta$, their sum and product are both $\frac{1}{a}$, so $(\alpha - 1)(\beta - 1) = 1$. Write

$$\alpha = 1 + \frac{1}{u}, \qquad \beta = 1 + u$$

with $u \geq 1$. Then $a = \dfrac{u}{(u+1)^2}$. The root $x = n\alpha$ lies in $[n, n+1)$ exactly when $n < u$, while $n\beta$ never lies there for a positive integer n.

The required positive total ensures $u > 1$. Let N be the largest positive integer less than u, so $N < u \leq N + 1$. The sum of all solutions is therefore

$$\frac{u+1}{u} \cdot \frac{N(N+1)}{2} = 420.$$

Because $\frac{u+1}{u}$ decreases with u, these inequalities imply

$$\frac{N(N+2)}{2} \leq 420 < \frac{(N+1)^2}{2},$$

which forces $N = 28$.

Substitution gives $406 \cdot \frac{u+1}{u} = 420$, so $u = 29$. Hence $a = \frac{29}{30^2} = \frac{29}{900}$. Indeed the positive solutions are $x = \frac{30n}{29}$ for $1 \leq n \leq 28$, and their sum is 420. Therefore $p + q = 29 + 900 = 929$.

Thus, **C** is the correct answer.

Problems: https://live.poshenloh.com/past-contests/amc12/2020A

2020 AMC 12B Solutions

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1. What is the value in simplest form of the following expression?

$$\sqrt{1}+\sqrt{1+3}+\sqrt{1+3+5}+\sqrt{1+3+5+7}$$

A 5

B $4+\sqrt{7}+\sqrt{10}$

C 10

D 15

E $4+3\sqrt{3}+2\sqrt{5}+\sqrt{7}$

Solution:

The sum of the first k odd numbers equals k^2, so each radicand is a perfect square:

$$\sqrt{1}+\sqrt{4}+\sqrt{9}+\sqrt{16}=1+2+3+4=10.$$

Thus, the correct answer is **C**.

2. What is the value of the following expression?

$$\frac{100^2 - 7^2}{70^2 - 11^2} \cdot \frac{(70 - 11)(70 + 11)}{(100 - 7)(100 + 7)}$$

A 1

B $\frac{9951}{9950}$

C $\frac{4780}{4779}$

D $\frac{108}{107}$

E $\frac{81}{80}$

Solution:

Using the difference of squares, $100^2 - 7^2 = (100 - 7)(100 + 7)$ and $70^2 - 11^2 = (70 - 11)(70 + 11)$. The expression becomes

$$\frac{(100 - 7)(100 + 7)}{(70 - 11)(70 + 11)} \cdot \frac{(70 - 11)(70 + 11)}{(100 - 7)(100 + 7)} = 1.$$

Thus, the correct answer is **A**.

3. The ratio of w to x is $4:3$, the ratio of y to z is $3:2$, and the ratio of z to x is $1:6$. What is the ratio of w to y?

A $4:3$

B $3:2$

C $8:3$

D $4:1$

E $16:3$

Solution:

Let $x = 6$. From $z:x = 1:6$, we get $z = 1$. From $w:x = 4:3$, we get $w = \frac{4}{3} \cdot 6 = 8$. From $y:z = 3:2$, we get $y = \frac{3}{2} \cdot 1 = \frac{3}{2}$.

Therefore $w:y = 8:\frac{3}{2} = 16:3$.

Thus, the correct answer is **E**.

4. The acute angles of a right triangle are a° and b°, where $a > b$ and both a and b are prime numbers. What is the least possible value of b?

A 2

B 3

C 5

D 7

E 11

Solution:

Since the angles are complementary, $a + b = 90$. To minimize b, try small primes and require $90 - b$ to be prime as well.

For $b = 2, 3, 5$, the value $90 - b = 88, 87, 85$ is not prime. For $b = 7$, we get $90 - 7 = 83$, which is prime. So the least possible value is $b = 7$.

Thus, the correct answer is **D**.

5. Teams A and B are playing in a basketball league where each game results in a win for one team and a loss for the other team. Team A has won $\frac{2}{3}$ of its games and team B has won $\frac{5}{8}$ of its games. Also, team B has won 7 more games and lost 7 more games than team A. How many games has team A played?

A 21

B 27

C 42

D 48

E 63

Solution:

Let a be the number of games team A played and b the number team B played. Team A wins $\frac{2}{3}a$ and loses $\frac{1}{3}a$; team B wins $\frac{5}{8}b$ and loses $\frac{3}{8}b$. The conditions give

$$\tfrac{5}{8}b = \tfrac{2}{3}a + 7$$

and

$$\tfrac{3}{8}b = \tfrac{1}{3}a + 7.$$

Subtracting the equations gives $\frac{1}{4}b = \frac{1}{3}a$, so $b = \frac{4}{3}a$. Substituting into the loss equation: $\frac{3}{8} \cdot \frac{4}{3}a = \frac{1}{3}a + 7$, i.e. $\frac{1}{2}a = \frac{1}{3}a + 7$, so $\frac{1}{6}a = 7$ and $a = 42$.

Thus, the correct answer is **C**.

6. For all integers $n \geq 9$, the value of

$$\frac{(n+2)! - (n+1)!}{n!}$$

is always which of the following?

A a multiple of 4

B a multiple of 10

C a prime number

D a perfect square

E a perfect cube

Solution:

Factor $(n+1)!$ from the numerator:

$$\begin{aligned} &(n+2)! - (n+1)! \\ &= (n+1)!\left[(n+2) - 1\right] \\ &= (n+1)!\,(n+1). \end{aligned}$$

Dividing by $n!$ leaves $\dfrac{(n+1)!\,(n+1)}{n!} = (n+1)(n+1) = (n+1)^2$, which is always a perfect square.

Thus, the correct answer is **D**.

7. Two nonhorizontal, non-vertical lines in the xy-coordinate plane intersect to form a 45° angle. One line has slope equal to 6 times the slope of the other line. What is the greatest possible value of the product of the slopes of the two lines?

A $\frac{1}{6}$

B $\frac{2}{3}$

C $\frac{3}{2}$

D 3

E 6

Solution:

Let the slopes be m and $6m$. The angle between the lines satisfies

$$\left|\frac{6m - m}{1 + 6m^2}\right| = \tan 45^\circ = 1,$$

so $5m = \pm(1 + 6m^2)$, giving $6m^2 - 5m + 1 = 0$ or $6m^2 + 5m + 1 = 0$.

The first yields $m = \frac{1}{2}$ or $m = \frac{1}{3}$; the second yields the negatives of these. The product of the slopes is $6m^2$, which is largest when $m = \frac{1}{2}$, giving $6 \cdot \frac{1}{4} = \frac{3}{2}$.

Thus, the correct answer is **C**.

8. How many ordered pairs of integers (x, y) satisfy the equation

$$x^{2020} + y^2 = 2y?$$

A 1

B 2

C 3

D 4

E infinitely many

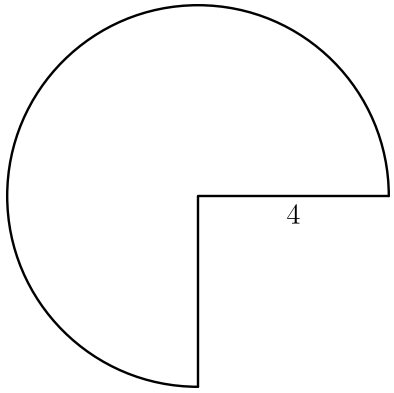
Solution:

Completing the square gives $x^{2020} + (y-1)^2 = 1$. Both terms are nonnegative, so $x^{2020} \leq 1$, forcing $x \in \{-1, 0, 1\}$.

If $x = 0$, then $(y-1)^2 = 1$, giving $y = 0$ or $y = 2$. If $x = \pm 1$, then $x^{2020} = 1$, so $(y-1)^2 = 0$ and $y = 1$. The solutions are $(0, 0)$, $(0, 2)$, $(1, 1)$, and $(-1, 1)$ — four in all.

Thus, the correct answer is **D**.

9. A three-quarter sector of a circle of radius 4 inches together with its interior can be rolled up to form the lateral surface of a right circular cone by taping together along the two radii shown. What is the volume of the cone in cubic inches?



A $3\pi\sqrt{5}$

B $4\pi\sqrt{3}$

C $3\pi\sqrt{7}$

D $6\pi\sqrt{3}$

E $6\pi\sqrt{7}$

Solution:

The sector's arc length is $\frac{3}{4} \cdot 2\pi \cdot 4 = 6\pi$, which becomes the base circumference: $2\pi r = 6\pi$, so $r = 3$.

The slant height is the sector radius 4, so the height is $h = \sqrt{4^2 - 3^2} = \sqrt{7}$. The volume is

$$\frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \cdot 9 \cdot \sqrt{7} = 3\pi\sqrt{7}.$$

Thus, the correct answer is **C**.

10. In unit square $ABCD$, the inscribed circle ω intersects $\overline{CD}$ at M, and $\overline{AM}$ intersects ω at a point P different from M. What is AP?

A $\frac{\sqrt{5}}{12}$

B $\frac{\sqrt{5}}{10}$

C $\frac{\sqrt{5}}{9}$

D $\frac{\sqrt{5}}{8}$

E $\frac{2\sqrt{5}}{15}$

Solution:

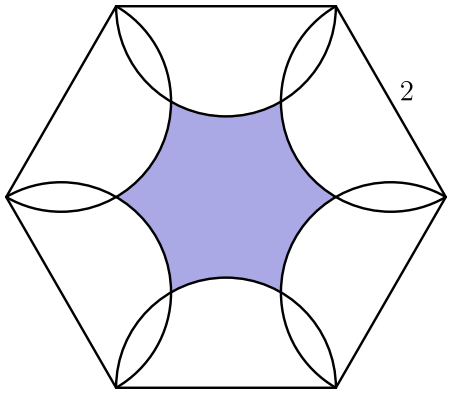
Let $A=(0,0), B=(1,0), C=(1,1), D=(0,1)$. The inscribed circle has center $\left(\frac{1}{2}, \frac{1}{2}\right)$ and radius $\frac{1}{2}$, touching $\overline{CD}$ at $M=\left(\frac{1}{2}, 1\right)$.

Line AM is $y=2x$. Substituting into $\left(x-\frac{1}{2}\right)^2+\left(y-\frac{1}{2}\right)^2=\frac{1}{4}$ gives $20x^2-12x+1=0$, with roots $x=\frac{1}{2}$ (point M) and $x=\frac{1}{10}$ (point P).

So $P=\left(\frac{1}{10}, \frac{1}{5}\right)$ and $AP=\sqrt{\left(\frac{1}{10}\right)^2+\left(\frac{1}{5}\right)^2}=\frac{\sqrt{5}}{10}$.

Thus, the correct answer is **B**.

11. As shown in the figure below, six semicircles lie in the interior of a regular hexagon with side length 2 so that the diameters of the semicircles coincide with the sides of the hexagon. What is the area of the shaded region—inside the hexagon but outside all of the semicircles?



A $6\sqrt{3} - 3\pi$

B $\frac{9\sqrt{3}}{2} - 2\pi$

C $\frac{3\sqrt{3}}{2} - \frac{\pi}{3}$

D $3\sqrt{3} - \pi$

E $\frac{9\sqrt{3}}{2} - \pi$

Solution:

The hexagon has area $\frac{3\sqrt{3}}{2} \cdot 2^2 = 6\sqrt{3}$. Each semicircle has radius 1 and area $\frac{\pi}{2}$, totaling 3π.

Adjacent semicircle centers (side midpoints) are a distance $\sqrt{3}$ apart, so each adjacent pair overlaps in a lens of area $2\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) - \frac{\sqrt{3}}{2} = \frac{\pi}{3} - \frac{\sqrt{3}}{2}$. There are six such lenses.

The union of the semicircles is

$$3\pi - 6\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right) = \pi + 3\sqrt{3}.$$

Subtracting from the hexagon gives the shaded area $6\sqrt{3} - (\pi + 3\sqrt{3}) = 3\sqrt{3} - \pi$.

Thus, the correct answer is **D**.

12. Let $\overline{AB}$ be a diameter in a circle of radius $5\sqrt{2}$. Let $\overline{CD}$ be a chord in the circle that intersects $\overline{AB}$ at a point E such that $BE = 2\sqrt{5}$ and $\angle AEC = 45^\circ$. What is $CE^2 + DE^2$?

A 96

B 98

C $44\sqrt{5}$

D $70\sqrt{2}$

E 100

Solution:

Place the center at the origin with $\overline{AB}$ on the x-axis; the radius is $R = 5\sqrt{2}$, so $R^2 = 50$. Then $E = (x_E, 0)$ with $x_E = R - 2\sqrt{5}$ (its exact value is not needed).

Parametrize the chord as $E + t\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. Substituting into $x^2 + y^2 = 50$ gives $t^2 + \sqrt{2}\, x_E\, t + (x_E^2 - 50) = 0$, whose roots are the signed distances t_1, t_2 to C and D.

By Vieta, $t_1 + t_2 = -\sqrt{2}\, x_E$ and $t_1 t_2 = x_E^2 - 50$, so

$$\begin{aligned} CE^2 + DE^2 &= t_1^2 + t_2^2 \\ &= (t_1 + t_2)^2 - 2t_1 t_2 \\ &= 2x_E^2 - 2(x_E^2 - 50) \\ &= 100. \end{aligned}$$

Thus, the correct answer is **E**.

13. Which of the following is the value of $\sqrt{\log_2 6 + \log_3 6}$?

A 1

B $\sqrt{\log_5 6}$

C 2

D $\sqrt{\log_2 3} + \sqrt{\log_3 2}$

E $\sqrt{\log_2 6} + \sqrt{\log_3 6}$

Solution:

Let $a = \log_2 3$, so $\log_3 2 = \frac{1}{a}$. Then

$$\begin{aligned} &\log_2 6 + \log_3 6 \\ &= (1 + \log_2 3) \\ &\quad + (1 + \log_3 2) \\ &= 2 + a + \frac{1}{a}. \end{aligned}$$

Meanwhile $\left(\sqrt{\log_2 3} + \sqrt{\log_3 2}\right)^2 = a + \frac{1}{a} + 2\sqrt{a \cdot \frac{1}{a}} = a + \frac{1}{a} + 2$, which equals the expression above.

Taking square roots, $\sqrt{\log_2 6 + \log_3 6} = \sqrt{\log_2 3} + \sqrt{\log_3 2}$.

Thus, the correct answer is **D**.

14. Bela and Jenn play the following game on the closed interval $[0, n]$ of the real number line, where n is a fixed integer greater than 4. They take turns playing, with Bela going first. At his first turn, Bela chooses any real number in the interval $[0, n]$. Thereafter, the player whose turn it is chooses a real number that is more than one unit away from all numbers previously chosen by either player. A player unable to choose such a number loses. Using optimal strategy, which player will win the game?

A Bela will always win.

B Jenn will always win.

C Bela will win if and only if n is odd.

D Jenn will win if and only if n is odd.

E Jenn will win if and only if $n > 8$.

Solution:

Bela first plays the midpoint $\frac{n}{2}$. This choice makes the configuration symmetric about the center of the interval.

Thereafter, whenever Jenn picks a number x, Bela responds with its mirror image $n - x$. Since Bela has already chosen $\frac{n}{2}$, Jenn's legal move satisfies $\left|x - \frac{n}{2}\right| > 1$. Therefore $|x - (n - x)| = 2\left|x - \frac{n}{2}\right| > 2$, so Bela's response is far enough from Jenn's new point. Symmetry shows that it is also far enough from every earlier point. Thus Bela always has a move whenever Jenn does, so Jenn is the first to be stuck. Bela always wins.

Thus, the correct answer is **A**.

15. There are 10 people standing equally spaced around a circle. Each person knows exactly 3 of the other 9 people: the 2 people standing next to her or him, as well as the person directly across the circle. How many ways are there for the 10 people to split up into 5 pairs so that the members of each pair know each other?

A 11

B 12

C 13

D 14

E 15

Solution:

Label the people 0 through 9. Allowed pairings use neighbor edges $(i, i+1)$ or diameter edges $(i, i+5)$. Count perfect matchings by the number of diameter edges used.

Using no diameters, the ten people split into adjacent pairs in 2 ways (all "even" edges or all "odd" edges). Using exactly one diameter, choose it in 5 ways; the remaining two arcs of four people each pair up uniquely, giving 5. Using all five diameters gives 1 matching.

If at least one diameter is used, following the forced adjacent pairings around the circle shows that the number of diameters must be odd. With exactly three diameters, the four remaining people must be two opposite adjacent pairs. The position of one such opposite pair determines the matching, giving 5 possibilities. Hence the total is $2 + 5 + 5 + 1 = 13$.

Thus, the correct answer is **C**.

16. An urn contains one red ball and one blue ball. A box of extra red and blue balls lies nearby. George performs the following operation four times: he draws a ball from the urn at random and then takes a ball of the same color from the box and returns those two matching balls to the urn. After the four iterations the urn contains six balls. What is the probability that the urn contains three balls of each color?

A $\frac{1}{6}$

B $\frac{1}{5}$

C $\frac{1}{4}$

D $\frac{1}{3}$

E $\frac{1}{2}$

Solution:

To end with three of each color, exactly two of the four added balls must be red. Consider any sequence of draws. When the urn holds k balls, drawing a particular color with count c has probability $\frac{c}{k}$.

Any ordering with two red and two blue additions gives the same product $\frac{1\cdot2\cdot1\cdot2}{2\cdot3\cdot4\cdot5}$, and there are $\binom{4}{2} = 6$ such orderings, for probability $\frac{6\cdot4}{120} = \frac{1}{5}$. (Equivalently, the number of red balls after four steps is uniform on $\{1, 2, 3, 4, 5\}$, so 3 red occurs with probability $\frac{1}{5}$.)

Thus, the correct answer is **B**.

17. How many polynomials of the form $x^5 + ax^4 + bx^3 + cx^2 + dx + 2020$, where $a, b, c,$ and d are real numbers, have the property that whenever r is a root, so is $\frac{-1 + i\sqrt{3}}{2} \cdot r$? (Note that $i = \sqrt{-1}$.)

A 0

B 1

C 2

D 3

E 4

Solution:

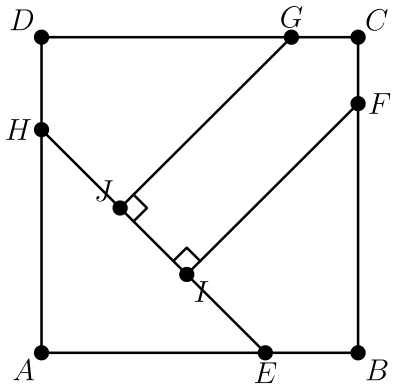
Here $\omega = \frac{-1+i\sqrt{3}}{2}$ is a primitive cube root of unity. Since 0 is not a root, the set of distinct roots is closed under multiplication by ω, so it consists of triples $\{r, \omega r, \omega^2 r\}$ equally spaced in argument. Five roots cannot fill two such triples, so there is exactly one triple, with multiplicities $m_1, m_2, m_3 \geq 1$ summing to 5.

Real coefficients require the root multiset to be closed under conjugation. This is possible only when the triple's arguments are symmetric about the real axis, which happens for the two configurations $\{0^\circ, 120^\circ, 240^\circ\}$ and $\{60^\circ, 180^\circ, 300^\circ\}$.

The product of the roots must equal -2020. In the first configuration the real root is positive, forcing a positive product, which is impossible. In the second, the real root is negative and the product is $-\rho^5$; setting $\rho^5 = 2020$ works, and the two conjugate-symmetric multiplicity patterns $(1, 3, 1)$ and $(2, 1, 2)$ each give a valid polynomial. Hence there are 2.

Thus, the correct answer is **C**.

18. In square $ABCD$, points E and H lie on $\overline{AB}$ and $\overline{DA}$, respectively, so that $AE = AH$. Points F and G lie on $\overline{BC}$ and $\overline{CD}$, respectively, and points I and J lie on $\overline{EH}$ so that $\overline{FI} \perp \overline{EH}$ and $\overline{GJ} \perp \overline{EH}$. See the figure below. Triangle AEH, quadrilateral $BFIE$, quadrilateral $DHJG$, and pentagon $FCGJI$ each has area 1. What is FI^2?



A $\frac{7}{3}$

B $8 - 4\sqrt{2}$

C $1 + \sqrt{2}$

D $\frac{7}{4}\sqrt{2}$

E $2\sqrt{2}$

Solution:

The four regions have total area 4, so the square has side 2. Put $A = (0,0), B = (2,0), C = (2,2), D = (0,2)$. Since $\triangle AEH$ is an isosceles right triangle with area 1, we get $AE = AH = \sqrt{2}$, so $E = (\sqrt{2}, 0)$ and $H = (0, \sqrt{2})$. Line EH is $x + y = \sqrt{2}$.

Let $F = (2, t)$. Its perpendicular distance to line EH is $FI = \frac{2+t-\sqrt{2}}{\sqrt{2}}$. Write $s = \frac{FI}{\sqrt{2}} = \frac{2+t-\sqrt{2}}{2}$, so the foot of the perpendicular is $I = (2-s, t-s)$. The shoelace formula on $B = (2,0), F, I$, and $E = (\sqrt{2}, 0)$ gives $[BFIE] = s^2 - (3 - 2\sqrt{2})$. Since this area is 1, we get $s^2 = 4 - 2\sqrt{2}$.

Then $FI^2 = 2s^2 = 8 - 4\sqrt{2}$.

Thus, the correct answer is **B**.

19. Square $ABCD$ in the coordinate plane has vertices at the points $A(1,1), B(-1,1), C(-1,-1)$, and $D(1,-1)$. Consider the following four transformations:

L, a rotation of 90° counterclockwise around the origin;

R, a rotation of 90° clockwise around the origin;

H, a reflection across the x-axis; and

V, a reflection across the y-axis.

Each of these transformations maps the square onto itself, but the positions of the labeled vertices will change. For example, applying R and then V would send the vertex A at $(1,1)$ to $(-1,-1)$ and would send the vertex B at $(-1,1)$ to itself. How many sequences of 20 transformations chosen from $\{L, R, H, V\}$ will send all of the labeled vertices back to their original positions? (For example, R, R, V, H is one sequence of 4 transformations that will send the vertices back to their original positions.)

A 2^{37}

B $3 \cdot 2^{36}$

C 2^{38}

D $3 \cdot 2^{37}$

E 2^{39}

Solution:

Label the vertices $0, 1, 2, 3$ cyclically. Each allowed transformation has the form $j \mapsto \varepsilon j + \delta \pmod 4$, where $\varepsilon \in \{1, -1\}$ and $\delta \in \{1, -1\}$. These four choices give exactly the two quarter-turns and the two stated reflections.

Under composition, the parity of δ changes at every move. Thus a composition of 19 allowed transformations again has odd δ, and so is one of the four allowed transformations. Its inverse is also allowed. Consequently every sequence of the first 19 moves has exactly one choice for the final move, giving $4^{19} = 2^{38}$ successful sequences.

Thus, the correct answer is **C**.

20. Two different cubes of the same size are to be painted, with the color of each face being chosen independently and at random to be either black or white. What is the probability that after they are painted, the cubes can be rotated to be identical in appearance?

A $\frac{9}{64}$

B $\frac{289}{2048}$

C $\frac{73}{512}$

D $\frac{147}{1024}$

E $\frac{589}{4096}$

Solution:

For a fixed first cube, the number of second cubes matching it (up to rotation) equals the size of its rotation orbit. So the desired probability is $\frac{1}{64^2}\sum_{\text{orbits}}(\text{orbit size})^2$.

Grouping by black-face count, the orbit sizes are: 0 or 6 black $\to$ 1; 1 or 5 black $\to$ 6; 2 or 4 black $\to$ 3 (opposite) and 12 (adjacent); 3 black $\to$ 8 (corner) and 12 (band). Then

$$\begin{aligned}\sum(\text{orbit size})^2 \\ = 1 + 36 + (9 + 144) \\ + (64 + 144) + (9 + 144) \\ + 36 + 1 = 588.\end{aligned}$$

The probability is $\frac{588}{4096} = \frac{147}{1024}$.

Thus, the correct answer is **D**.

21. How many positive integers n satisfy

$$\frac{n+1000}{70} = \lfloor \sqrt{n} \rfloor ?$$

(Recall that $\lfloor x \rfloor$ is the greatest integer not exceeding x.)

A 2

B 4

C 6

D 30

E 32

Solution:

The right side is an integer, so let $k = \lfloor \sqrt{n} \rfloor$. Then $n = 70k - 1000$, and $k = \lfloor \sqrt{n} \rfloor$ requires $k^2 \leq n < (k+1)^2$.

The lower bound $k^2 \leq 70k - 1000$ gives $k^2 - 70k + 1000 \leq 0$, i.e. $20 \leq k \leq 50$. The upper bound $70k - 1000 < (k+1)^2$ gives $k^2 - 68k + 1001 > 0$, i.e. $k \leq 21$ or $k \geq 47$.

Intersecting, $k \in \{20, 21, 47, 48, 49, 50\}$, giving 6 values of n.

Thus, the correct answer is **C**.

22. What is the maximum value of

$$\frac{(2^t - 3t)\,t}{4^t}$$

for real values of t?

A $\frac{1}{16}$

B $\frac{1}{15}$

C $\frac{1}{12}$

D $\frac{1}{10}$

E $\frac{1}{9}$

Solution:

Split the fraction: $\frac{(2^t - 3t)t}{4^t} = \frac{t}{2^t} - \frac{3t^2}{4^t}$. Let $u = \frac{t}{2^t}$, so $\frac{t^2}{4^t} = u^2$ and the expression is $u - 3u^2$.

This parabola has maximum at $u = \frac{1}{6}$, with value $\frac{1}{6} - 3 \cdot \frac{1}{36} = \frac{1}{12}$. Since $u = \frac{t}{2^t}$ is continuous and attains the value $\frac{1}{6}$, the maximum is achieved.

Thus, the correct answer is **C**.

23. How many integers $n \geq 2$ are there such that whenever $z_1, z_2, \ldots, z_n$ are complex numbers such that

$$|z_1| = |z_2| = \cdots = |z_n| = 1$$

and

$$z_1 + z_2 + \cdots + z_n = 0,$$

then the numbers $z_1, z_2, \ldots, z_n$ are equally spaced on the unit circle in the complex plane?

A 1

B 2

C 3

D 4

E 5

Solution:

For $n = 2$, $z_1 + z_2 = 0$ forces $z_2 = -z_1$, which is equally spaced. For $n = 3$, three unit vectors summing to zero must form an equilateral triangle, so they are equally spaced.

For every even $n \geq 4$, choose $\frac{n}{2}$ antipodal pairs at generic angles that do not form a regular n-gon. For every odd $n \geq 5$, choose the vertices of an equilateral triangle together with $\frac{n-3}{2}$ generic antipodal pairs. Each construction has sum 0 but is not equally spaced.

Hence only $n = 2$ and $n = 3$ work, giving 2 values.

Thus, the correct answer is **B**.

24. Let $D(n)$ denote the number of ways of writing the positive integer n as a product

$$n = f_1 \cdot f_2 \cdots f_k,$$

where $k \geq 1$, the f_i are integers strictly greater than 1, and the order in which the factors are listed matters (that is, two representations that differ only in the order of the factors are counted as distinct). For example, the number 6 can be written as 6, $2 \cdot 3$, and $3 \cdot 2$, so $D(6) = 3$. What is $D(96)$?

A 112

B 128

C 144

D 172

E 184

Solution:

The first factor f_1 can be any divisor $d > 1$, after which the rest is an ordered factorization of $\frac{n}{d}$. So $D(n) = \sum_{d|n,\, d>1} D(\frac{n}{d})$, with $D(1) = 1$.

Computing over the divisors of $96 = 2^5 \cdot 3$: $D(2) = D(3) = 1, D(4) = 2, D(6) = 3, D(8) = 4, D(12) = 8, D(16) = 8, D(24) = 20, D(32) = 16, D(48) = 48.$

Finally

$$\begin{aligned} D(96) &= D(48) + D(32) \\ &+ D(24) + D(16) + D(12) \\ &+ D(8) + D(6) + D(4) \\ &+ D(3) + D(2) + D(1) \\ &= 48 + 16 + 20 + 8 \\ &+ 8 + 4 + 3 + 2 \\ &+ 1 + 1 + 1 = 112. \end{aligned}$$

Thus, the correct answer is **A**.

25. For each real number a with $0 \le a \le 1$, let numbers x and y be chosen independently at random from the intervals $[0, a]$ and $[0, 1]$, respectively, and let $P(a)$ be the probability that

$$\sin^2(\pi x) + \sin^2(\pi y) > 1.$$

What is the maximum value of $P(a)$?

A $\dfrac{7}{12}$

B $2 - \sqrt{2}$

C $\dfrac{1 + \sqrt{2}}{4}$

D $\dfrac{\sqrt{5} - 1}{2}$

E $\dfrac{5}{8}$

Solution:

Since $\sin^2(\pi y) = 1 - \cos^2(\pi y)$, the condition is $|\sin \pi x| > |\cos \pi y|$. For fixed x, the probability over $y \in [0, 1]$ is $g(x) = 2x$ for $0 \le x \le \frac{1}{2}$ and $g(x) = 2 - 2x$ for $\frac{1}{2} \le x \le 1$.

Then $P(a) = \frac{1}{a} \int_0^a g(x)\, dx$. For $a \le \frac{1}{2}$, $P(a) = a$, increasing to $\frac{1}{2}$. For $a \ge \frac{1}{2}$,

$$\begin{aligned} P(a) &= \frac{2a - a^2 - \frac{1}{2}}{a} \\ &= 2 - a - \frac{1}{2a}. \end{aligned}$$

Setting the derivative to zero gives $2a^2 = 1$, so $a = \frac{1}{\sqrt{2}}$, and $P\left(\frac{1}{\sqrt{2}}\right) = 2 - \sqrt{2}$.

Thus, the correct answer is **B**.

Problems: https://live.poshenloh.com/past-contests/amc12/2020B