AMC 12 SOLUTION ARCHIVE

2019

12A and 12B full explanations

Contents

AMC 12A full solutions	26 pages	**starts p. 2**
AMC 12B full solutions	28 pages	**starts p. 28**

About these explanations

Each section contains a complete problem-by-problem solution booklet curated by LIVE by Po-Shen Loh. The archive states that the AMC problems are used with official legal permission from the Mathematical Association of America.

Open AMC 12A solution archive

Open AMC 12B solution archive

Solutions source: LIVE by Po-Shen Loh. AMC problems copyright MAA.

Compiled 2026-08-17

2019 AMC 12A Solutions

Typeset by: LIVE by Po-Shen Loh
https://live.poshenloh.com/past-contests/amc12/2019A/solutions

Problems © Mathematical Association of America. Reproduced with permission.

1. The area of a pizza with radius 4 inches is N percent larger than the area of a pizza with radius 3 inches. What is the integer closest to N?

A 25

B 33

C 44

D 66

E 78

Solution:

The areas are proportional to the squares of the radii, so the ratio of the larger area to the smaller is $\frac{16}{9}$.

The percent increase is

$$\left(\frac{16}{9}-1\right)\times 100=\frac{700}{9}\approx 77.8.$$

The closest integer is 78.

Thus, the correct answer is **E**.

2. Suppose a is 150% of b. What percent of a is $3b$?

A 50

B $66\frac{2}{3}$

C 150

D 200

E 450

Solution:

Since $a = 1.5b$, we have

$$\frac{3b}{a} = \frac{3b}{1.5b} = 2.$$

As a percentage, $3b$ is 200% of a.

Thus, the correct answer is **D**.

3. A box contains 28 red balls, 20 green balls, 19 yellow balls, 13 blue balls, 11 white balls, and 9 black balls. What is the minimum number of balls that must be drawn from the box without replacement to guarantee that at least 15 balls of a single color will be drawn?

A 75

B 76

C 79

D 84

E 91

Solution:

In the worst case, we draw 14 each of red, green, and yellow, plus all of the blue (13), white (11), and black (9), without reaching 15 of any color.

That is

$$14 + 14 + 14 + 13 + 11 + 9 = 75$$

balls.

The next ball must complete a set of 15, so 76 balls are needed.

Thus, the correct answer is **B**.

4. What is the greatest number of consecutive integers whose sum is 45?

A 9

B 25

C 45

D 90

E 120

Solution:

Negative integers are allowed. The integers from -44 to 44 sum to 0, so the integers from -44 to 45 sum to 45.

This run has $45 - (-44) + 1 = 90$ integers. Conversely, if L consecutive integers have sum 45, then twice their sum is L times an integer, so $L \mid 90$. Hence $L \leq 90$, proving that this run is longest.

Thus, the correct answer is **D**.

5. Two lines with slopes $\frac{1}{2}$ and 2 intersect at $(2, 2)$. What is the area of the triangle enclosed by these two lines and the line $x + y = 10$?

A 4

B $4\sqrt{2}$

C 6

D 8

E $6\sqrt{2}$

Solution:

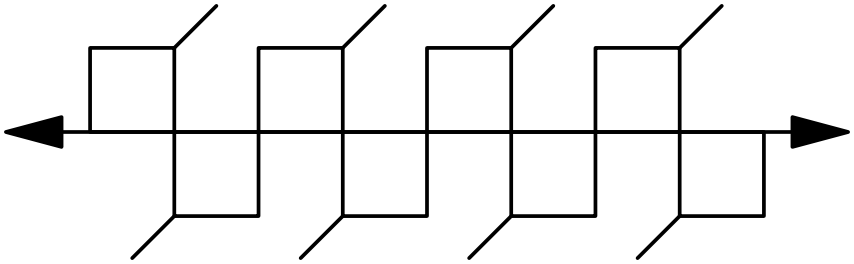
The two lines are $y = \frac{1}{2}x + 1$ and $y = 2x - 2$. Intersecting each with $x + y = 10$ gives the points $(6, 4)$ and $(4, 6)$.

The triangle has vertices $(2, 2)$, $(6, 4)$, and $(4, 6)$. By the shoelace formula,

$$\begin{aligned} &\tfrac{1}{2}\left|2(4-6) + 6(6-2) + 4(2-4)\right| \\ &= \tfrac{1}{2}\left|-4 + 24 - 8\right| \\ &= 6. \end{aligned}$$

Thus, the correct answer is **C**.

6. The figure below shows line ℓ with a regular, infinite, recurring pattern of squares and line segments.



How many of the following four kinds of rigid motion transformations of the plane in which this figure is drawn, other than the identity transformation, will transform this figure into itself?

- some rotation around a point of line ℓ
- some translation in the direction parallel to line ℓ
- the reflection across line ℓ
- some reflection across a line perpendicular to line ℓ

A 0

B 1

C 2

D 3

E 4

Solution:

A translation by one full period maps the figure to itself, so translation works.

A 180° rotation about a suitable point on ℓ sends each square above the line to the square below it, with the diagonal segments matching, so this rotation works.

Reflection across ℓ sends the top-right diagonals to top-right diagonals below the line, but the actual below-line diagonals point to the bottom-left, so it fails. A

reflection across a perpendicular line fails for the same reason. Only 2 of the four transformations work.

Thus, the correct answer is **C**.

7. Melanie computes the mean μ, the median M, and the modes of the 365 values that are the dates in the months of 2019. Thus her data consist of 12 1s, 12 2s, $\ldots$, 12 28s, 11 29s, 11 30s, and 7 31s. Let d be the median of the modes. Which of the following statements is true?

A $\mu < d < M$

B $M < d < \mu$

C $d = M = \mu$

D $d < M < \mu$

E $d < \mu < M$

Solution:

The values 1 through 28 each appear 12 times and are the modes, so $d = \dfrac{14 + 15}{2} = 14.5$.

The 183rd of the 365 ordered values is the median. Values 1 through 15 fill the first 180 positions, so position 183 is 16; thus $M = 16$.

The total of all values is $12(1 + \cdots + 28) + 11(29 + 30) + 7 \cdot 31 = 5738$, so $\mu = \dfrac{5738}{365} \approx 15.7$.

Therefore $d < \mu < M$.

Thus, the correct answer is **E**.

8. For a set of four distinct lines in a plane, there are exactly N distinct points that lie on two or more of the lines. What is the sum of all possible values of N?

A 14

B 16

C 18

D 19

E 21

Solution:

The values 0 and 1 occur when all four lines are parallel or all four are concurrent. Three parallel lines crossed by a fourth give 3 points. Three concurrent lines together with a fourth line not through their common point give 4. One parallel pair with no three concurrent gives 5, and four lines in general position give 6.

The value 2 is impossible. Once two lines meet at P, a third line not through P creates at least one new intersection; a fourth distinct line must then create another new point. If every remaining line passes through P, there is only one intersection point instead. Thus the achievable values are $0, 1, 3, 4, 5, 6$, whose sum is 19.

Thus, the correct answer is **D**.

9. A sequence of numbers is defined recursively by $a_1 = 1, a_2 = \frac{3}{7}$, and

$$a_n = \frac{a_{n-2} \cdot a_{n-1}}{2a_{n-2} - a_{n-1}}$$

for all $n \geq 3$. Then a_{2019} can be written as $\frac{p}{q}$, where p and q are relatively prime positive integers. What is $p + q$?

A 2020

B 4039

C 6057

D 6061

E 8078

Solution:

Taking reciprocals,

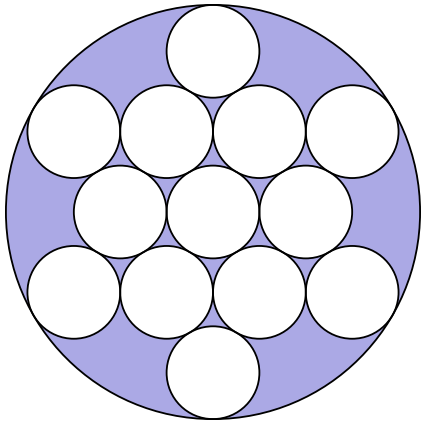
$$\begin{aligned} \frac{1}{a_n} &= \frac{2a_{n-2} - a_{n-1}}{a_{n-2}a_{n-1}} \\ &= \frac{2}{a_{n-1}} - \frac{1}{a_{n-2}}. \end{aligned}$$

Let $b_n = \frac{1}{a_n}$. Then $b_n = 2b_{n-1} - b_{n-2}$, so b_n is arithmetic with $b_1 = 1, b_2 = \frac{7}{3}$, and common difference $\frac{4}{3}$.

Thus $b_{2019} = 1 + 2018 \cdot \frac{4}{3} = \frac{8075}{3}$, so $a_{2019} = \frac{3}{8075}$. Since these are relatively prime, $p + q = 8078$.

Thus, the correct answer is **E**.

10. The figure below shows 13 circles of radius 1 within a larger circle. All the intersections occur at points of tangency. What is the area of the region, shaded in the figure, inside the larger circle but outside all the circles of radius 1?



A $4\pi\sqrt{3}$

B 7π

C $\pi(3\sqrt{3}+2)$

D $10\pi(\sqrt{3}-1)$

E $\pi(\sqrt{3}+6)$

Solution:

Place a unit circle at the center, six around it with centers at distance 2 (a hexagon), and six more with centers at distance $2\sqrt{3}$ in the outer gaps. That is $1+6+6=13$ circles.

The outermost circles are tangent to the big circle, whose radius is therefore $2\sqrt{3}+1$. Its area is

$$\pi(2\sqrt{3}+1)^2=\pi(13+4\sqrt{3}).$$

Subtracting the 13 unit circles leaves $\pi(13+4\sqrt{3})-13\pi=4\pi\sqrt{3}$.

Thus, the correct answer is **A**.

11. For some positive integer k, the repeating base-k representation of the (base-ten) fraction $\frac{7}{51}$ is $0.\overline{23}_k = 0.232323\ldots_k$. What is k?

A 13

B 14

C 15

D 16

E 17

Solution:

The repeating block gives

$$0.\overline{23}_k = \frac{2k+3}{k^2-1} = \frac{7}{51}.$$

Cross-multiplying, $51(2k+3) = 7(k^2-1)$, so $7k^2 - 102k - 160 = 0$.

The quadratic formula gives

$$\begin{aligned} k &= \frac{102 + \sqrt{14884}}{14} \\ &= \frac{102+122}{14} = 16. \end{aligned}$$

Thus, the correct answer is **D**.

12. Positive real numbers $x \neq 1$ and $y \neq 1$ satisfy $\log_2 x = \log_y 16$ and $xy = 64$. What is $\left(\log_2 \dfrac{x}{y}\right)^2$?

A $\dfrac{25}{2}$

B 20

C $\dfrac{45}{2}$

D 25

E 32

Solution:

Let $a = \log_2 x$ and $b = \log_2 y$. Then $\log_y 16 = \dfrac{4}{b}$, so $a = \dfrac{4}{b}$, giving $ab = 4$.

Since $xy = 64$, we have $a + b = 6$.

Therefore

$$\begin{aligned}\left(\log_2 \tfrac{x}{y}\right)^2 &= (a-b)^2 \\ &= (a+b)^2 - 4ab \\ &= 36 - 16 = 20.\end{aligned}$$

Thus, the correct answer is **B**.

13. How many ways are there to paint each of the integers $2, 3, \ldots, 9$ either red, green, or blue so that each number has a different color from each of its proper divisors?

A 144

B 216

C 256

D 384

E 432

Solution:

The primes 5 and 7 have no proper divisors here, giving 3 choices each.

Along the chain $2 \to 4 \to 8$, there are $3 \cdot 2 \cdot 1 = 6$ colorings. Number 9 must differ from 3, giving 2 choices once 3 is set.

Number 6 must differ from both 2 and 3. Summing over the colors of 2 and 3 (equal in 3 pairs, unequal in 6 pairs), the combined factor for $4, 8, 9, 6$ totals $2 \cdot 2 \cdot (3 \cdot 2 + 6 \cdot 1) = 48$.

Multiplying by the 9 ways for 5 and 7 gives $48 \cdot 9 = 432$.

Thus, the correct answer is **E**.

14. For a certain complex number c, the polynomial

$$P(x) = (x^2 - 2x + 2) \cdot (x^2 - cx + 4) \cdot (x^2 - 4x + 8)$$

has exactly 4 distinct roots. What is $|c|$?

A 2

B $\sqrt{6}$

C $2\sqrt{2}$

D 3

E $\sqrt{10}$

Solution:

The factors $x^2 - 2x + 2$ and $x^2 - 4x + 8$ have roots $1 \pm i$ and $2 \pm 2i$, which are 4 distinct values.

For P to have exactly 4 distinct roots, the roots of $x^2 - cx + 4$ must lie among these. Their product must equal 4, and the only such pair is one root from each factor, for example $(1 + i)(2 - 2i) = 4$.

Then $c = (1 + i) + (2 - 2i) = 3 - i$, so $|c| = \sqrt{3^2 + 1^2} = \sqrt{10}$.

Thus, the correct answer is **E**.

15. Positive real numbers a and b have the property that

$$\sqrt{\log a} + \sqrt{\log b} + \log \sqrt{a} + \log \sqrt{b} = 100$$

and all four terms on the left are positive integers, where $\log$ denotes the base 10 logarithm. What is ab?

A 10^{52}

B 10^{100}

C 10^{144}

D 10^{164}

E 10^{200}

Solution:

Let $\sqrt{\log a} = p$ and $\sqrt{\log b} = q$, so $\log a = p^2$ and $\log \sqrt{a} = \frac{p^2}{2}$. For this to be an integer, p is even; likewise q.

Writing $p = 2m, q = 2n$, the equation $p + q + \frac{p^2}{2} + \frac{q^2}{2} = 100$ becomes $m(m+1) + n(n+1) = 50$.

The only solution is $\{m, n\} = \{4, 5\}$, giving $\log(ab) = p^2 + q^2 = 4(16 + 25) = 164$.

Therefore $ab = 10^{164}$.

Thus, the correct answer is **D**.

16. The numbers $1, 2, \ldots, 9$ are randomly placed into the 9 squares of a 3×3 grid. Each square gets one number, and each of the numbers is used once. What is the probability that the sum of the numbers in each row and each column is odd?

A $\frac{1}{21}$

B $\frac{1}{14}$

C $\frac{5}{63}$

D $\frac{2}{21}$

E $\frac{1}{7}$

Solution:

There are 5 odd and 4 even numbers. Each row and column must contain an odd number of odd entries.

The only way to place 5 odd entries with every row and column odd is to fill one complete row and one complete column (a plus shape of $3 + 3 - 1 = 5$ cells). There are $3 \cdot 3 = 9$ such patterns.

Each pattern admits $5!$ placements of the odd numbers and $4!$ of the even numbers, so the probability is

$$\frac{9 \cdot 5! \cdot 4!}{9!} = \frac{1}{14}.$$

Thus, the correct answer is **B**.

17. Let s_k denote the sum of the kth powers of the roots of the polynomial $x^3 - 5x^2 + 8x - 13$. In particular, $s_0 = 3$, $s_1 = 5$, and $s_2 = 9$. Let a, b, and c be real numbers such that $s_{k+1} = a\,s_k + b\,s_{k-1} + c\,s_{k-2}$ for $k = 2, 3, \ldots$. What is $a + b + c$?

A -6

B 0

C 6

D 10

E 26

Solution:

Every root r satisfies $r^3 = 5r^2 - 8r + 13$, so $r^{k+1} = 5r^k - 8r^{k-1} + 13r^{k-2}$.

Summing over the three roots gives $s_{k+1} = 5s_k - 8s_{k-1} + 13s_{k-2}$, so $a = 5, b = -8, c = 13$.

Therefore $a + b + c = 5 - 8 + 13 = 10$.

Thus, the correct answer is **D**.

18. A sphere with center O has radius 6. A triangle with sides of length $15, 15$, and 24 is situated in space so that each of its sides is tangent to the sphere. What is the distance between O and the plane determined by the triangle?

A $2\sqrt{3}$

B 4

C $3\sqrt{2}$

D $2\sqrt{5}$

E 5

Solution:

The sphere intersects the triangle's plane in a circle of radius $\sqrt{36 - d^2}$, where d is the distance from O to the plane. Since each side is tangent to the sphere, this circle is the triangle's incircle.

The triangle has area $\frac{1}{2} \cdot 24 \cdot 9 = 108$ and semiperimeter 27, so its inradius is $\frac{108}{27} = 4$.

Thus $\sqrt{36 - d^2} = 4$, giving $d^2 = 20$ and $d = 2\sqrt{5}$.

Thus, the correct answer is **D**.

19. In $\triangle ABC$ with integer side lengths,

$$\cos A = \frac{11}{16},$$
$$\cos B = \frac{7}{8},$$
$$\cos C = -\frac{1}{4}.$$

What is the least possible perimeter for $\triangle ABC$?

A 9

B 12

C 23

D 27

E 44

Solution:

Each sine is $\sqrt{1-\cos^2}$: $\sin A = \frac{3\sqrt{15}}{16}, \sin B = \frac{2\sqrt{15}}{16}, \sin C = \frac{4\sqrt{15}}{16}$.

By the Law of Sines the sides are in ratio $3:2:4$. The smallest integer sides are $3, 2, 4$, which satisfy the triangle inequality.

The least perimeter is $3+2+4=9$.

Thus, the correct answer is **A**.

20. Real numbers between 0 and 1, inclusive, are chosen in the following manner. A fair coin is flipped. If it lands heads, then it is flipped again and the chosen number is 0 if the second flip is heads and 1 if the second flip is tails. On the other hand, if the first coin flip is tails, then the number is chosen uniformly at random from the closed interval $[0, 1]$. Two random numbers x and y are chosen independently in this manner. What is the probability that $|x - y| > \frac{1}{2}$?

A $\frac{1}{3}$

B $\frac{7}{16}$

C $\frac{1}{2}$

D $\frac{9}{16}$

E $\frac{2}{3}$

Solution:

Each variable equals 0 with probability $\frac{1}{4}$, equals 1 with probability $\frac{1}{4}$, and is uniform on $[0, 1]$ with probability $\frac{1}{2}$.

Considering the nine combinations of types: the pairs $(0, 1)$ and $(1, 0)$ each contribute $\frac{1}{16}$. Each of the four point-versus-uniform cases contributes $\frac{1}{16}$. The uniform-versus-uniform case contributes $\frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$.

The total is $\frac{2 + 4 + 1}{16} = \frac{7}{16}$.

Thus, the correct answer is **B**.

21. Let

$$z = \frac{1+i}{\sqrt{2}}.$$

What is

$$\left(z^{1^2} + z^{2^2} + z^{3^2} + \cdots + z^{12^2}\right) \cdot \left(\frac{1}{z^{1^2}} + \frac{1}{z^{2^2}} + \frac{1}{z^{3^2}} + \cdots + \frac{1}{z^{12^2}}\right)?$$

A 18

B $72 - 36\sqrt{2}$

C 36

D 72

E $72 + 36\sqrt{2}$

Solution:

Since $z = e^{\frac{i\pi}{4}}$, we have $z^{k^2} = e^{\frac{i\pi k^2}{4}}$, depending only on $k^2 \bmod 8$.

For $k = 1$ to 12, the residue $k^2 \bmod 8$ is 1 (giving z) six times, 4 (giving -1) three times, and 0 (giving 1) three times. So the first sum is $6z - 3 + 3 = 6z$.

The second sum is likewise $\frac{6}{z} - 3 + 3 = \frac{6}{z}$. Their product is $6z \cdot \frac{6}{z} = 36$.

Thus, the correct answer is **C**.

22. Circles ω and γ, both centered at O, have radii 20 and 17, respectively. Equilateral triangle ABC, whose interior lies in the interior of ω but in the exterior of γ, has vertex A on ω, and the line containing side BC is tangent to γ. Segments AO and BC intersect at P, and $\frac{BP}{CP} = 3$. Then AB can be written in the form $\frac{m}{\sqrt{n}} - \frac{p}{\sqrt{q}}$ for positive integers m, n, p, q with $\gcd(m, n) = \gcd(p, q) = 1$. What is $m + n + p + q$?

A 42

B 86

C 92

D 114

E 130

Solution:

Let $s = AB$. Since $\frac{BP}{CP} = 3$, we have $BP = \frac{3s}{4}$ and $CP = \frac{s}{4}$. Put P at the origin with BC on the x-axis, $B = \left(-\frac{3s}{4}, 0\right), C = \left(\frac{s}{4}, 0\right)$, and apex $A = \left(-\frac{s}{4}, \frac{s\sqrt{3}}{2}\right)$.

Points P, O, A are collinear, so $O = t \cdot A$ for some scalar t. Two conditions pin it down: O is at distance 17 from line BC, giving $|t| \cdot \frac{s\sqrt{3}}{2} = 17$, and A is on ω, giving $|t - 1| \cdot \frac{s\sqrt{13}}{4} = 20$ since $|A| = \frac{s\sqrt{13}}{4}$.

Solving, $|t|s = \frac{34}{\sqrt{3}}$ and $|t - 1|s = \frac{80}{\sqrt{13}}$. The valid configuration has O and A on opposite sides of P, so $|t - 1|s = |t|s + s$. Therefore

$$AB = s = \frac{80}{\sqrt{13}} - \frac{34}{\sqrt{3}}.$$

Then $m + n + p + q = 80 + 13 + 34 + 3 = 130$.

Thus, the correct answer is **E**.

23. Define binary operations $\diamondsuit$ and $\heartsuit$ by

$$a \diamondsuit b = a^{\log_7(b)}$$

and

$$a \heartsuit b = a^{\frac{1}{\log_7(b)}}$$

for all real numbers a and b for which these expressions are defined. The sequence (a_n) is defined recursively by $a_3 = 3 \heartsuit 2$ and

$$a_n = (n \heartsuit (n-1)) \diamondsuit a_{n-1}$$

for all integers $n \geq 4$. To the nearest integer, what is $\log_7(a_{2019})$?

A 8

B 9

C 10

D 11

E 12

Solution:

Let $L(x) = \log_7 x$. Then $L(a \diamondsuit b) = L(a)L(b)$ and $L(a \heartsuit b) = \dfrac{L(a)}{L(b)}$.

So $L(a_3) = \dfrac{L(3)}{L(2)}$, and $L(a_n) = \dfrac{L(n)}{L(n-1)} \cdot L(a_{n-1})$. The product telescopes:

$$\begin{aligned} L(a_N) &= \frac{L(3)}{L(2)} \cdot \frac{L(N)}{L(3)} \\ &= \frac{L(N)}{L(2)}. \end{aligned}$$

Hence $L(a_{2019}) = \dfrac{\log_7 2019}{\log_7 2} = \log_2 2019 \approx 10.98$, which rounds to 11.

Thus, the correct answer is **D**.

24. For how many integers n between 1 and 50, inclusive, is

$$\frac{(n^2-1)!}{(n!)^n}$$

an integer? (Recall that $0! = 1$.)

A 31

B 32

C 33

D 34

E 35

Solution:

Fix a prime $p \le n$. By Legendre's formula, the difference between the exponent of p in the numerator and its exponent in the denominator is

$$D_p = \sum_{k\ge 1} \left\lfloor \frac{n^2-1}{p^k} \right\rfloor - n \sum_{k\ge 1} \left\lfloor \frac{n}{p^k} \right\rfloor.$$

If r_k is the remainder of n modulo p^k, the kth summand is $\left\lfloor \dfrac{nr_k - 1}{p^k} \right\rfloor$.

Let $a = v_p(n)$. The first a summands are -1. If n is not a power of p, write $n = p^a m$ with $m \ge 2$. When $a \ge 1$, the next summand is at least $\dfrac{n}{p} - 1 \ge a$, and all later summands are nonnegative; when $a = 0$, every summand is already nonnegative. Thus $D_p \ge 0$ unless n is a power of p.

For $n = p^a$, Legendre's formula reduces the requirement to $p^a - 1 \ge 2a(p-1)$. Among prime powers at most 50, this fails exactly when $a = 1$ (so n is prime) and when $n = 2^2 = 4$. There are 15 primes at most 50, plus $n = 4$, giving 16 failures. Hence $50 - 16 = 34$ values of n work.

Thus, the correct answer is **D**.

25. Let $\triangle A_0B_0C_0$ be a triangle whose angle measures are exactly 59.999°, 60°, and 60.001°. For each positive integer n define A_n to be the foot of the altitude from A_{n-1} to line $B_{n-1}C_{n-1}$. Likewise, define B_n to be the foot of the altitude from B_{n-1} to line $A_{n-1}C_{n-1}$, and C_n to be the foot of the altitude from C_{n-1} to line $A_{n-1}B_{n-1}$. What is the least positive integer n for which $\triangle A_nB_nC_n$ is obtuse?

A 10

B 11

C 13

D 14

E 15

Solution:

For an acute triangle, the orthic triangle (feet of the altitudes) has angles $180^\circ - 2\alpha$ for each original angle α.

Writing an angle as $60^\circ + x$, the new angle is $60^\circ - 2x$, so each deviation from 60° is multiplied by -2. The initial deviations are $\pm 0.001^\circ$.

After n steps a deviation has magnitude $0.001 \cdot 2^n$ degrees. The triangle first becomes obtuse when this exceeds 30°, i.e. $2^n > 30000$. Since $2^{14} = 16384$ and $2^{15} = 32768$, the least such n is 15.

Thus, the correct answer is **E**.

Problems: https://live.poshenloh.com/past-contests/amc12/2019A

2019 AMC 12B Solutions

Typeset by: LIVE by Po-Shen Loh
https://live.poshenloh.com/past-contests/amc12/2019B/solutions

Problems © Mathematical Association of America. Reproduced with permission.

1. Alicia had two containers. The first was $\frac{5}{6}$ full of water and the second was empty. She poured all the water from the first container into the second container, at which point the second container was $\frac{3}{4}$ full of water. What is the ratio of the volume of the first container to the volume of the second container?

A $\frac{5}{8}$

B $\frac{4}{5}$

C $\frac{7}{8}$

D $\frac{9}{10}$

E $\frac{11}{12}$

Solution:

The volume of water is the same before and after, so $\frac{5}{6}V_1 = \frac{3}{4}V_2$.

Then

$$\frac{V_1}{V_2} = \frac{\frac{3}{4}}{\frac{5}{6}} = \frac{3}{4} \cdot \frac{6}{5} = \frac{9}{10}.$$

Thus, **D** is the correct answer.

2. Consider the statement, "If n is not prime, then $n-2$ is prime." Which of the following values of n is a counterexample to this statement?

A 11

B 15

C 19

D 21

E 27

Solution:

A counterexample needs n not prime (so the hypothesis holds) and $n-2$ not prime (so the conclusion fails).

Among the choices, 27 is not prime and $27-2=25$ is not prime. The primes 11 and 19 fail the hypothesis, and $15-2=13, 21-2=19$ are prime.

Thus, **E** is the correct answer.

3. Which one of the following rigid transformations (isometries) maps the line segment $\overline{AB}$ onto the line segment $\overline{A'B'}$ so that the image of $A(-2, 1)$ is $A'(2, -1)$ and the image of $B(-1, 4)$ is $B'(1, -4)$?

A reflection in the y-axis

B counterclockwise rotation around the origin by 90°

C translation by 3 units to the right and 5 units down

D reflection in the x-axis

E clockwise rotation about the origin by 180°

Solution:

Each point maps by $(x, y) \to (-x, -y)$: indeed $(-2, 1) \to (2, -1)$ and $(-1, 4) \to (1, -4)$.

The map $(x, y) \to (-x, -y)$ is a 180° rotation about the origin.

Thus, **E** is the correct answer.

4. A positive integer n satisfies the equation $(n+1)! + (n+2)! = 440 \cdot n!$. What is the sum of the digits of n?

A 2

B 5

C 10

D 12

E 15

Solution:

Factor the left side:

$$\begin{aligned} &(n+1)! + (n+2)! \\ &= (n+1)!\,[1 + (n+2)] \\ &= (n+1)!\,(n+3). \end{aligned}$$

Dividing both sides by $n!$ and using $(n+1)! = (n+1)\,n!$ gives

$$(n+1)(n+3) = 440.$$

So $n^2 + 4n - 437 = 0$, which factors as $(n-19)(n+23) = 0$, giving $n = 19$. Its digit sum is $1 + 9 = 10$.

Thus, **C** is the correct answer.

5. Each piece of candy in a store costs a whole number of cents. Casper has exactly enough money to buy either 12 pieces of red candy, 14 pieces of green candy, 15 pieces of blue candy, or n pieces of purple candy. A piece of purple candy costs 20 cents. What is the smallest possible value of n?

A 18

B 21

C 24

D 25

E 28

Solution:

Let M be Casper's money in cents. Since he can exactly buy $12, 14,$ or 15 whole-cent pieces, M is a multiple of $\text{lcm}(12, 14, 15) = 420.$

Purple candy costs 20 cents, so $n = \frac{M}{20}$. The smallest M is 420, giving $n = \frac{420}{20} = 21.$

Thus, **B** is the correct answer.

6. In a given plane, points A and B are 10 units apart. How many points C are there in the plane such that the perimeter of $\triangle ABC$ is 50 units and the area of $\triangle ABC$ is 100 square units?

A 0

B 2

C 4

D 8

E infinitely many

Solution:

The perimeter condition gives $CA + CB = 50 - 10 = 40$, so C lies on an ellipse with foci A, B and major axis $2a = 40$. Thus $a = 20$ and $c = 5$, so the semi-minor axis is

$$b = \sqrt{a^2 - c^2} = \sqrt{375} \approx 19.36.$$

For area 100 with base $AB = 10$, the height from C must be $\dfrac{2 \cdot 100}{10} = 20$. But the greatest possible height on the ellipse is $b \approx 19.36 < 20$, so no such C exists.

Thus, **A** is the correct answer.

7. What is the sum of all real numbers x for which the median of the numbers $4, 6, 8, 17,$ and x is equal to the mean of those five numbers?

A -5

B 0

C 5

D $\frac{15}{4}$

E $\frac{35}{4}$

Solution:

The mean is $\frac{4+6+8+17+x}{5} = \frac{35+x}{5}$.

If $x \leq 6$, the median is 6, so $\frac{35+x}{5} = 6$ gives $x = -5$, which is consistent.

If $6 < x < 8$, the median is x, so $\frac{35+x}{5} = x$ gives $x = 8.75$, not in range. If $x \geq 8$, the median is 8, so $\frac{35+x}{5} = 8$ gives $x = 5$, not in range.

The only solution is $x = -5$, so the sum is -5.

Thus, **A** is the correct answer.

8. Let $f(x) = x^2(1-x)^2$. What is the value of the sum

$$f\left(\tfrac{1}{2019}\right) - f\left(\tfrac{2}{2019}\right) + f\left(\tfrac{3}{2019}\right) - f\left(\tfrac{4}{2019}\right) + \cdots + f\left(\tfrac{2017}{2019}\right) - f\left(\tfrac{2018}{2019}\right)?$$

A 0

B $\dfrac{1}{2019^4}$

C $\dfrac{2018^2}{2019^4}$

D $\dfrac{2020^2}{2019^4}$

E 1

Solution:

Since $f(1-x) = (1-x)^2x^2 = f(x)$, we have $f\left(\tfrac{k}{2019}\right) = f\left(\tfrac{2019-k}{2019}\right)$.

In the sum, the term with index k has sign $(-1)^{k+1}$, while the term with index $2019-k$ equals it in value but has sign $(-1)^{2019-k+1} = (-1)^k$, the opposite.

Every term cancels with its partner, so the total is 0.

Thus, **A** is the correct answer.

9. For how many integral values of x can a triangle of positive area be formed having side lengths $\log_2 x, \log_4 x$, and 3?

A 57

B 59

C 61

D 62

E 63

Solution:

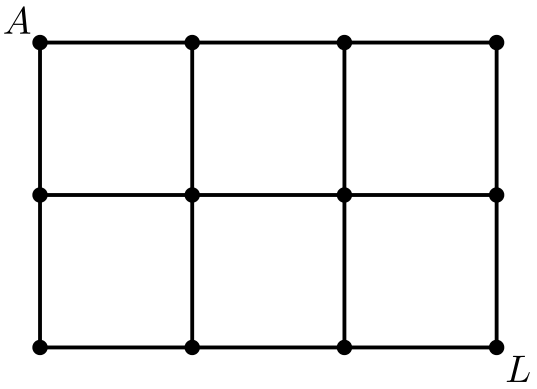
Let $t = \log_2 x$. Then $\log_4 x = \frac{t}{2}$, and the sides are $t,\ \frac{t}{2},\ 3$.

The triangle inequalities give $t + \frac{t}{2} > 3$ (so $t > 2$) and $\frac{t}{2} + 3 > t$ (so $t < 6$); the third inequality is automatic.

Thus $2 < \log_2 x < 6$, i.e. $4 < x < 64$. The integers $5, 6, \ldots, 63$ number 59.

Thus, **B** is the correct answer.

10. The figure below is a map showing 12 cities and 17 roads connecting certain pairs of cities. Paula wishes to travel along exactly 13 of those roads, starting at city A and ending at city L, without traveling along any portion of a road more than once. (Paula is allowed to visit a city more than once.) How many different routes can Paula take?



A 0

B 1

C 2

D 3

E 4

Solution:

Name the four cities in the top row A, B, C, D, those in the middle row E, F, G, H, and those in the bottom row I, J, K, L. A route using 13 roads is an open Euler trail, so in the used graph exactly A and L have odd degree.

In the full map, the vertices whose degree parity must change are A, B, C, E, H, J, K, L. Because only 4 roads are removed, those roads must pair these 8 vertices. Among them, E is adjacent only to A, forcing AE to be removed; then BC is forced. Similarly H is adjacent only to L, forcing HL, and then JK.

The remaining graph is a chain

$$A - B - F,$$
$$F - E - I - J - F,$$
$$F - G,$$
$$G - C - D - H - G,$$
$$G - K - L.$$

Each of the two 4-cycles can be traversed in either direction, and everything else is forced. Hence there are $2 \cdot 2 = 4$ routes.

Thus, **E** is the correct answer.

11. How many unordered pairs of edges of a given cube determine a plane?

A 12

B 28

C 36

D 42

E 66

Solution:

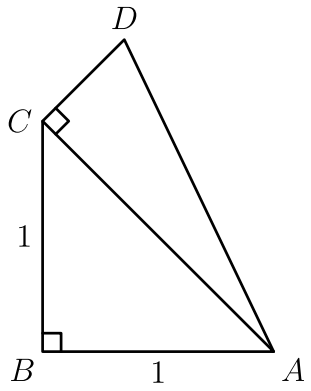
Two edges determine a plane exactly when they are coplanar, that is, parallel or intersecting.

The 12 edges split into 3 directions of 4 parallel edges, giving $3\binom{4}{2} = 18$ parallel pairs. Edges sharing a vertex give $8\binom{3}{2} = 24$ intersecting pairs.

The total is $18 + 24 = 42$.

Thus, **D** is the correct answer.

12. Right triangle ACD with right angle at C is constructed outwards on the hypotenuse $\overline{AC}$ of isosceles right triangle ABC with leg length 1, as shown, so that the two triangles have equal perimeters. What is $\sin(2\angle BAD)$?



A $\frac{1}{3}$

B $\frac{\sqrt{2}}{2}$

C $\frac{3}{4}$

D $\frac{7}{9}$

E $\frac{\sqrt{3}}{2}$

Solution:

Triangle ABC has perimeter $1+1+\sqrt{2}=2+\sqrt{2}$ and $AC=\sqrt{2}$. In $\triangle ACD$ let $CD=d$, so $AD=\sqrt{2+d^2}$ and equal perimeters give

$$\sqrt{2}+d+\sqrt{2+d^2}=2+\sqrt{2}.$$

Then $\sqrt{2+d^2}=2-d$, so $2+d^2=4-4d+d^2$, giving $d=\frac{1}{2}$ and $AD=\frac{3}{2}$.

Since $\angle BAC=45^\circ$, writing $\theta=\angle CAD$ gives $2\angle BAD=90^\circ+2\theta$, so $\sin(2\angle BAD)=\cos 2\theta$. With $\tan\theta=\frac{CD}{AC}=\frac{1}{2\sqrt{2}}$, we get

$$\begin{aligned}\cos 2\theta &= \frac{1-\tan^2\theta}{1+\tan^2\theta} \\ &= \frac{1-\frac{1}{8}}{1+\frac{1}{8}} = \frac{7}{9}.\end{aligned}$$

Thus, **D** is the correct answer.

13. A red ball and a green ball are randomly and independently tossed into bins numbered with the positive integers so that for each ball, the probability that it is tossed into bin k is 2^{-k} for $k = 1, 2, 3, \ldots$ What is the probability that the red ball is tossed into a higher-numbered bin than the green ball?

A $\frac{1}{4}$

B $\frac{2}{7}$

C $\frac{1}{3}$

D $\frac{3}{8}$

E $\frac{3}{7}$

Solution:

The probability the balls land in the same bin is

$$\sum_{k=1}^{\infty} \left(2^{-k}\right)^2 = \sum_{k=1}^{\infty} 4^{-k}$$
$$= \frac{\frac{1}{4}}{1 - \frac{1}{4}} = \frac{1}{3}.$$

By symmetry, the red ball being higher and the green ball being higher are equally likely, so each has probability

$$\frac{1 - \frac{1}{3}}{2} = \frac{1}{3}.$$

Thus, **C** is the correct answer.

14. Let S be the set of all positive integer divisors of $100{,}000$. How many numbers are the product of two distinct elements of S?

A 98

B 100

C 117

D 119

E 121

Solution:

Since $100{,}000 = 2^5 \cdot 5^5$, every divisor is $2^a 5^b$ with $0 \le a, b \le 5$. A product of two divisors is $2^x 5^y$ with $0 \le x, y \le 10$, and every such pair (x, y) is attainable, giving $11 \cdot 11 = 121$ values.

We need two *distinct* divisors. A value $2^x 5^y$ is forced to be a divisor times itself only when both x and y have a unique split, which happens exactly when $x, y \in \{0, 10\}$. Those 4 corner values ($1,\ 2^{10},\ 5^{10},\ 2^{10}5^{10}$) cannot use two distinct divisors.

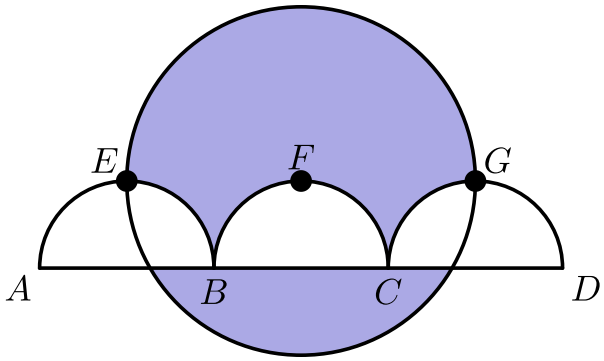
The count is $121 - 4 = 117$.

Thus, **C** is the correct answer.

15. As shown in the figure, line segment $\overline{AD}$ is trisected by points B and C so that $AB = BC = CD = 2$. Three semicircles of radius 1, AEB, BFC, and CGD, have their diameters on $\overline{AD}$, and are tangent to line EG at E, F, and G, respectively. A circle of radius 2 has its center on F. The area of the region inside the circle but outside the three semicircles, shaded in the figure, can be expressed in the form

$$\frac{a}{b} \cdot \pi - \sqrt{c} + d,$$

where a, b, c, and d are positive integers and a and b are relatively prime. What is $a + b + c + d$?



A 13

B 14

C 15

D 16

E 17

Solution:

Put $A = (0,0),\ B = (2,0),\ C = (4,0),\ D = (6,0)$, so the semicircles are centered at $(1,0), (3,0), (5,0)$ and their tops are $E = (1,1),\ F = (3,1),\ G = (5,1)$. The circle has center $F = (3,1)$ and radius 2, so it passes through E and G, and has area 4π.

The middle semicircle BFC lies entirely inside the circle, removing area $\frac{\pi}{2}$. For the left semicircle, the overlap starts at E and includes its right-hand quarter-circle, except for the region below the large circle. The large circle meets the x-axis at $x = 3 - \sqrt{3}$. The excluded area is

$$\begin{aligned} I &= \int_1^{3-\sqrt{3}} 1 \, dx \\ &\quad - \int_1^{3-\sqrt{3}} \sqrt{4 - (x-3)^2} \, dx \\ &= 2 - \frac{\sqrt{3}}{2} - \frac{\pi}{3}. \end{aligned}$$

Therefore the overlap has area

$$\begin{aligned} R &= \frac{\pi}{4} - I \\ &= \frac{7\pi}{12} - 2 + \frac{\sqrt{3}}{2}. \end{aligned}$$

By symmetry, the right semicircle contributes the same overlap.

The shaded area is

$$4\pi - \frac{\pi}{2} - 2R = \frac{7}{3}\pi - \sqrt{3} + 4.$$

Hence $a = 7,\ b = 3,\ c = 3,\ d = 4$, so $a + b + c + d = 17$.

Thus, **E** is the correct answer.

16. There are lily pads in a row numbered 0 to 11, in that order. There are predators on lily pads 3 and 6, and a morsel of food on lily pad 10. Fiona the frog starts on pad 0, and from any given lily pad, has a $\frac{1}{2}$ chance to hop to the next pad, and an equal chance to jump 2 pads. What is the probability that Fiona reaches pad 10 without landing on either pad 3 or pad 6?

A $\frac{15}{256}$

B $\frac{1}{16}$

C $\frac{15}{128}$

D $\frac{1}{8}$

E $\frac{1}{4}$

Solution:

Let $p(n)$ be the probability of landing on pad n without first landing on pad 3 or 6. Each pad sends probability $\frac{1}{2}$ to the next pad and $\frac{1}{2}$ two pads ahead, and pads 3 and 6 pass nothing on.

Then $p(0) = 1$, $p(1) = \frac{1}{2}$, $p(2) = \frac{3}{4}$, and (skipping 3) $p(4) = \frac{3}{8}$, $p(5) = \frac{3}{16}$, then (skipping 6) $p(7) = \frac{3}{32}$, $p(8) = \frac{3}{64}$, $p(9) = \frac{9}{128}$.

Finally

$$\begin{aligned} p(10) &= \frac{1}{2}p(8) + \frac{1}{2}p(9) \\ &= \frac{3}{128} + \frac{9}{256} \\ &= \frac{15}{256}. \end{aligned}$$

Thus, **A** is the correct answer.

17. How many nonzero complex numbers z have the property that 0, z, and z^3, when represented by points in the complex plane, are the three distinct vertices of an equilateral triangle?

A 0

B 1

C 2

D 4

E infinitely many

Solution:

The three points form an equilateral triangle iff $|z| = |z^3| = |z^3 - z|$. From $|z| = |z^3| = |z|^3$ we get $|z| = 1$.

Then $|z^3 - z| = |z|\,|z^2 - 1| = |z^2 - 1|$, so we need $|z^2 - 1| = 1$. Writing $z = e^{i\theta}$, $|z^2 - 1| = 2|\sin\theta| = 1$, so $|\sin\theta| = \dfrac{1}{2}$.

This gives $\theta = 30^\circ, 150^\circ, 210^\circ, 330^\circ$, four values of z, all yielding distinct vertices.

Thus, **D** is the correct answer.

18. Square pyramid $ABCDE$ has base $ABCD$, which measures 3 cm on a side, and altitude $\overline{AE}$ perpendicular to the base, which measures 6 cm. Point P lies on $\overline{BE}$, one third of the way from B to E; point Q lies on $\overline{DE}$, one third of the way from D to E; and point R lies on $\overline{CE}$, two thirds of the way from C to E. What is the area, in square centimeters, of $\triangle PQR$?

A $\dfrac{3\sqrt{2}}{2}$

B $\dfrac{3\sqrt{3}}{2}$

C $2\sqrt{2}$

D $2\sqrt{3}$

E $3\sqrt{2}$

Solution:

Place $A = (0,0,0)$, $B = (3,0,0)$, $C = (3,3,0)$, $D = (0,3,0)$, $E = (0,0,6)$. Then

$$P = (2,0,2),$$
$$Q = (0,2,2),$$
$$R = (1,1,4).$$

So $\vec{PQ} = (-2,2,0)$ and $\vec{PR} = (-1,1,2)$, giving $\vec{PQ} \times \vec{PR} = (4,4,0)$.

The area is $\dfrac{1}{2}\left|(4,4,0)\right| = \dfrac{1}{2} \cdot 4\sqrt{2} = 2\sqrt{2}$.

Thus, **C** is the correct answer.

19. Raashan, Sylvia, and Ted play the following game. Each starts with $\$1$. A bell rings every 15 seconds, at which time each of the players who currently have money simultaneously chooses one of the other two players independently and at random and gives $\$1$ to that player. What is the probability that after the bell has rung 2019 times, each player will have $\$1$? (For example, Raashan and Ted may each decide to give $\$1$ to Sylvia, and Sylvia may decide to give her dollar to Ted, at which point Raashan will have $\$0$, Sylvia will have $\$2$, and Ted will have $\$1$, and that is the end of the first round of play. In the second round Raashan has no money to give, but Sylvia and Ted might choose each other to give their $\$1$ to, and the holdings will be the same at the end of the second round.)

A $\frac{1}{7}$

B $\frac{1}{4}$

C $\frac{1}{3}$

D $\frac{1}{2}$

E $\frac{2}{3}$

Solution:

From $(1,1,1)$, each of the three players gives to one of two others, so there are 8 equally likely outcomes; only the 2 cyclic gift patterns return to $(1,1,1)$, a probability of $\frac{1}{4}$.

From a $(2,1,0)$ state the broke player gives nothing, and checking the 4 equally likely choices of the other two shows exactly one yields $(1,1,1)$, again probability $\frac{1}{4}$.

So after any ring the probability of $(1,1,1)$ is $\frac{1}{4}$, including after 2019 rings.

Thus, **B** is the correct answer.

20. Points $A(6, 13)$ and $B(12, 11)$ lie on circle ω in the plane. Suppose that the tangent lines to ω at A and B intersect at a point on the x-axis. What is the area of ω?

A $\frac{83\pi}{8}$

B $\frac{21\pi}{2}$

C $\frac{85\pi}{8}$

D $\frac{43\pi}{4}$

E $\frac{87\pi}{8}$

Solution:

Let $P = (x, 0)$ be the intersection. Equal tangent lengths give $PA = PB$, so $(x - 6)^2 + 13^2 = (x - 12)^2 + 11^2$, yielding $x = 5$ and $P = (5, 0)$.

The center $O = (h, k)$ satisfies $OA \perp PA$ and $OB \perp PB$. With $PA = (1, 13)$ and $PB = (7, 11)$, these give $h + 13k = 175$ and $7h + 11k = 205$, so $O = \left(\frac{37}{4}, \frac{51}{4}\right)$.

Then $r^2 = OA^2 = \left(\frac{13}{4}\right)^2 + \left(\frac{1}{4}\right)^2 = \frac{170}{16} = \frac{85}{8}$, so the area is $\frac{85\pi}{8}$.

Thus, **C** is the correct answer.

21. How many quadratic polynomials with real coefficients are there such that the set of roots equals the set of coefficients? (For clarification: If the polynomial is $ax^2 + bx + c, a \neq 0$, and the roots are r and s, then the requirement is that $\{a, b, c\} = \{r, s\}$.)

A 3

B 4

C 5

D 6

E infinitely many

Solution:

If all three coefficients had one value u, the polynomial would be $u(x^2 + x + 1)$, whose roots do not equal u. Thus the coefficient and root sets both have two distinct values, so exactly two coefficients coincide. By Vieta's formulas, $r + s = -\frac{b}{a}$ and $rs = \frac{c}{a}$.

First suppose $a = b = u$ and $c = v$. The roots are u, v, so Vieta gives $u + v = -1$ and $uv = \frac{v}{u}$. Hence $v(u^2 - 1) = 0$, producing $x^2 + x - 2$ and $-x^2 - x$.

If $b = c = v$ and $a = u$, the same product equation gives $v(u^2 - 1) = 0$, while the sum equation leaves only $u = 1,\ v = -\frac{1}{2}$. This gives $x^2 - \frac{1}{2}x - \frac{1}{2}$.

Finally, if $a = c = u$ and $b = v$, the product equation gives $uv = 1$, so $v = \frac{1}{u}$. The sum equation becomes $u^3 + u + 1 = 0$. This strictly increasing cubic has one real root u, producing exactly one more polynomial, $ux^2 + \frac{1}{u}x + u$. Therefore there are 4 polynomials.

Thus, **B** is the correct answer.

22. Define a sequence recursively by $x_0 = 5$ and

$$x_{n+1} = \frac{x_n^2 + 5x_n + 4}{x_n + 6}$$

for all nonnegative integers n. Let m be the least positive integer such that

$$x_m \leq 4 + \frac{1}{2^{20}}.$$

In which of the following intervals does m lie?

A $[9, 26]$

B $[27, 80]$

C $[81, 242]$

D $[243, 728]$

E $[729, \infty)$

Solution:

Let $a_n = x_n - 4$. A short computation gives

$$\begin{aligned} a_{n+1} &= x_{n+1} - 4 \\ &= \frac{(x_n + 5)(x_n - 4)}{x_n + 6} \\ &= a_n \cdot \frac{x_n + 5}{x_n + 6}. \end{aligned}$$

Starting from $a_0 = 1$, the terms stay positive and decrease. Because x_n decreases from 5 toward 4, each ratio $\dfrac{x_n + 5}{x_n + 6}$ lies strictly between $\dfrac{9}{10}$ and $\dfrac{10}{11}$.

Hence a_m is squeezed between $\left(\dfrac{9}{10}\right)^m$ and $\left(\dfrac{10}{11}\right)^m$. Solving $a_m \leq 2^{-20}$ puts m between about 132 and 146, which lies in $[81, 242]$.

Thus, **C** is the correct answer.

23. How many sequences of 0s and 1s of length 19 are there that begin with a 0, end with a 0, contain no two consecutive 0s, and contain no three consecutive 1s?

A 55

B 60

C 65

D 70

E 75

Solution:

No two 0s are adjacent, so the 0s are separated by blocks of 1s, each of size 1 or 2 (never 3). If there are k zeros, there are $k-1$ such blocks summing to $19-k$ ones.

The number of size-2 blocks is $(19-k)-(k-1)=20-2k$, which must satisfy $0 \le 20-2k \le k-1$, i.e. $7 \le k \le 10$.

Summing $\binom{k-1}{20-2k}$ over $k=7,8,9,10$ gives $\binom{6}{6}+\binom{7}{4}+\binom{8}{2}+\binom{9}{0}=1+35+28+1=65$.

Thus, **C** is the correct answer.

24. Let $\omega = -\frac{1}{2} + \frac{1}{2}i\sqrt{3}$. Let S denote all points in the complex plane of the form $a + b\omega + c\omega^2$, where $0 \leq a \leq 1, 0 \leq b \leq 1$, and $0 \leq c \leq 1$. What is the area of S?

A $\frac{1}{2}\sqrt{3}$

B $\frac{3}{4}\sqrt{3}$

C $\frac{3}{2}\sqrt{3}$

D $\frac{1}{2}\pi\sqrt{3}$

E π

Solution:

As a, b, c range over $[0, 1]$, the set S is the Minkowski sum of the three unit segments along $v_1 = 1 = (1, 0)$, $v_2 = \omega = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, $v_3 = \omega^2 = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$.

This is a zonogon whose area is the sum of the cross-product magnitudes over pairs.

Each pair gives $|v_i \times v_j| = \frac{\sqrt{3}}{2}$.

Therefore the area is $3 \cdot \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$.

Thus, **C** is the correct answer.

25. Let $ABCD$ be a convex quadrilateral with $BC = 2$ and $CD = 6$. Suppose that the centroids of $\triangle ABC$, $\triangle BCD$, and $\triangle ACD$ form the vertices of an equilateral triangle. What is the maximum possible value of the area of $ABCD$?

A 27

B $16\sqrt{3}$

C $12 + 10\sqrt{3}$

D $9 + 12\sqrt{3}$

E 30

Solution:

The centroids are $\frac{A+B+C}{3}$, $\frac{B+C+D}{3}$, $\frac{A+C+D}{3}$. Their pairwise differences are $\frac{A-D}{3}$, $\frac{B-A}{3}$, $\frac{B-D}{3}$, so an equilateral centroid triangle forces $AB = BD = DA$; that is, $\triangle ABD$ is equilateral with side $s = BD$.

Splitting along BD,

$$\begin{aligned}[ABCD] &= [ABD] + [BCD] \\ &= \frac{\sqrt{3}}{4}s^2 \\ &\quad + \frac{1}{2} \cdot 2 \cdot 6 \sin C,\end{aligned}$$

where $C = \angle BCD$. By the Law of Cosines $s^2 = 40 - 24\cos C$, so

$$\begin{aligned}[ABCD] &= 10\sqrt{3} \\ &\quad - 6\sqrt{3}\cos C + 6\sin C.\end{aligned}$$

The expression $6\sin C - 6\sqrt{3}\cos C$ has maximum $\sqrt{6^2 + (6\sqrt{3})^2} = 12$, so the greatest area is $10\sqrt{3} + 12 = 12 + 10\sqrt{3}$. Equality occurs at $C = 150^\circ$; constructing $\triangle BCD$ with that angle and placing equilateral $\triangle ABD$ on the opposite side of $\overline{BD}$ produces a convex quadrilateral, so the maximum is attainable.

Thus, **C** is the correct answer.

Problems: https://live.poshenloh.com/past-contests/amc12/2019B