

AMC 12 SOLUTION ARCHIVE

2018

12A and 12B full explanations

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2018 AMC 12A Solutions

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1. A large urn contains 100 balls, of which 36% are red and the rest are blue. How many of the blue balls must be removed so that the percentage of red balls in the urn will be 72%? (No red balls are to be removed.)

A 28

B 32

C 36

D 50

E 64

Solution:

There are 36 red balls, and this count stays fixed. For the red balls to be 72% of the urn, the urn must contain $36 \div 0.72 = 50$ balls. Since $100 - 50 = 50$, exactly 50 blue balls are removed.

Thus, the correct answer is **D**.

2. While exploring a cave, Carl comes across a collection of 5-pound rocks worth \$14 each, 4-pound rocks worth \$11 each, and 1-pound rocks worth \$2 each. There are at least 20 of each size. He can carry at most 18 pounds. What is the maximum value, in dollars, of the rocks he can carry out of the cave?

A 48

B 49

C 50

D 51

E 52

Solution:

The rocks are worth \$2.80, \$2.75, and \$2 per pound, respectively. For each possible number of 5-pound rocks, use as many 4-pound rocks as fit and fill any leftover capacity with 1-pound rocks. Taking 0, 1, 2, or 3 of the 5-pound rocks gives maximum values \$48, \$49, \$50, and \$48, respectively.

Thus two 5-pound and two 4-pound rocks use all 18 pounds, and the maximum value is \$50. Thus, the correct answer is **C**.

3. How many ways can a student schedule 3 mathematics courses—algebra, geometry, and number theory—in a 6-period day if no two mathematics courses can be taken in consecutive periods? (What courses the student takes during the other 3 periods is of no concern here.)

A 3

B 6

C 12

D 18

E 24

Solution:

The choices of three non-consecutive periods are $\{1, 3, 5\}$, $\{1, 3, 6\}$, $\{1, 4, 6\}$, and $\{2, 4, 6\}$, a total of 4. The three distinct courses can be placed into any such set in $3! = 6$ orders, giving $4 \cdot 6 = 24$ schedules.

Thus, the correct answer is **E**.

4. Alice, Bob, and Charlie were on a hike and were wondering how far away the nearest town was. When Alice said, “We are at least 6 miles away,” Bob replied, “We are at most 5 miles away.” Charlie then remarked, “Actually the nearest town is at most 4 miles away.” It turned out that none of the three statements was true. Let d be the distance in miles to the nearest town. Which of the following intervals is the set of all possible values of d ?

A $(0, 4)$

B $(4, 5)$

C $(4, 6)$

D $(5, 6)$

E $(5, \infty)$

Solution:

Negating the three false statements gives $d < 6$, $d > 5$, and $d > 4$. The intersection of these conditions is $5 < d < 6$, that is, the interval $(5, 6)$.

Thus, the correct answer is **D**.

5. What is the sum of all possible values of k for which the polynomials $x^2 - 3x + 2$ and $x^2 - 5x + k$ have a root in common?

A 3

B 4

C 5

D 6

E 10

Solution:

Since $x^2 - 3x + 2 = (x - 1)(x - 2)$, its roots are 1 and 2. If 1 is a shared root then $1 - 5 + k = 0$, so $k = 4$. If 2 is a shared root then $4 - 10 + k = 0$, so $k = 6$. The sum of possible values is $4 + 6 = 10$.

Thus, the correct answer is **E**.

6. For positive integers m and n such that $m + 10 < n + 1$, both the mean and the median of the set $\{m, m + 4, m + 10, n + 1, n + 2, 2n\}$ are equal to n . What is $m + n$?

A 20

B 21

C 22

D 23

E 24

Solution:

Because $m + 10 < n + 1$, the six numbers are already increasing, so the median is the average of the middle two: $\frac{(m+10)+(n+1)}{2} = n$, giving $m = n - 11$. The mean condition is

$$\frac{(n - 11) + (n - 7) + (n - 1) + (n + 1) + (n + 2) + 2n}{6} = n,$$

so $7n - 16 = 6n$ and $n = 16$. Then $m = 5$, and $m + n = 21$.

Thus, the correct answer is **B**.

7. For how many (not necessarily positive) integer values of n is the value of

$$4000 \cdot \left(\frac{2}{5}\right)^n$$

an integer?

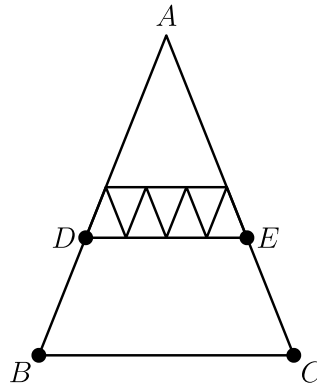
- A 3
- B 4
- C 6
- D 8
- E 9**

Solution:

Since $4000 = 2^5 \cdot 5^3$, the expression equals $2^{5+n} \cdot 5^{3-n}$. This is an integer exactly when both $5 + n \geq 0$ and $3 - n \geq 0$, that is, $-5 \leq n \leq 3$. There are $3 - (-5) + 1 = 9$ such integers.

Thus, the correct answer is **E**.

8. All of the triangles in the diagram below are similar to isosceles triangle ABC , in which $AB = AC$. Each of the 7 smallest triangles has area 1, and $\triangle ABC$ has area 40. What is the area of trapezoid $DBCE$?



- ☐ A 16
- ☐ B 18
- ☐ C 20
- ☐ D 22
- ☒ E 24

Solution:

The base DE of $\triangle ADE$ is 4 times the base of a smallest triangle, so by the square scaling of similar areas, $[ADE] = 4^2 \cdot 1 = 16$. The trapezoid $DBCE$ is what remains of $\triangle ABC$, so its area is $40 - 16 = 24$.

Thus, the correct answer is **E**.

9. Which of the following describes the largest subset of values of y within the closed interval $[0, \pi]$ for which

$$\sin(x + y) \leq \sin(x) + \sin(y)$$

for every x between 0 and π , inclusive?

A $y = 0$

B $0 \leq y \leq \frac{\pi}{4}$

C $0 \leq y \leq \frac{\pi}{2}$

D $0 \leq y \leq \frac{3\pi}{4}$

E $0 \leq y \leq \pi$

Solution:

For $0 \leq x \leq \pi$ and $0 \leq y \leq \pi$ we have $\sin x \geq 0$, $\sin y \geq 0$, $\cos x \leq 1$, and $\cos y \leq 1$. Hence

$$\begin{aligned}\sin(x + y) &= \sin x \cos y \\ &\quad + \cos x \sin y \\ &\leq \sin x + \sin y.\end{aligned}$$

The inequality therefore holds for every y with $0 \leq y \leq \pi$.

Thus, the correct answer is **E**.

10. How many ordered pairs of real numbers (x, y) satisfy the following system of equations?

$$x + 3y = 3$$

$$| |x| - |y| | = 1$$

A 1

B 2

C 3

D 4

E 8

Solution:

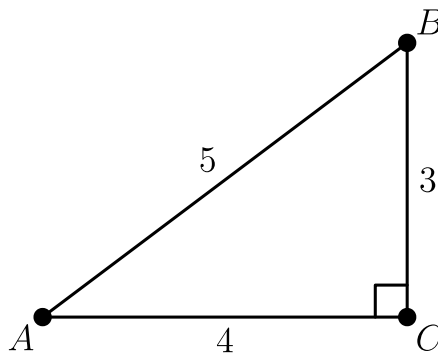
The second equation gives $|x| - |y| = \pm 1$, equivalently $x = \pm y \pm 1$. Substituting into $x + 3y = 3$:

If $x = y + 1$, then $(x, y) = \left(\frac{3}{2}, \frac{1}{2}\right)$. If $x = y - 1$, then $(x, y) = (0, 1)$. If $x = -y + 1$, then again $(x, y) = (0, 1)$. If $x = -y - 1$, then $(x, y) = (-3, 2)$.

The distinct solutions are $(-3, 2)$, $(0, 1)$, and $\left(\frac{3}{2}, \frac{1}{2}\right)$, all of which check, so there are 3.

Thus, the correct answer is **C**.

11. A paper triangle with sides of lengths 3, 4, and 5 inches, as shown, is folded so that point A falls on point B . What is the length in inches of the crease?



- ☐ A $1 + \frac{1}{2}\sqrt{2}$
- ☐ B $\sqrt{3}$
- ☐ C $\frac{7}{4}$
- ☒ D $\frac{15}{8}$
- ☐ E 2

Solution:

The crease lies along the perpendicular bisector of AB , meeting AC at E because $AC > BC$. Let D be the midpoint of AB , so $AD = \frac{5}{2}$ and $\triangle ADE$ is right-angled at D . Since $\triangle ADE \sim \triangle ACB$, we have $\frac{DE}{AD} = \frac{CB}{AC} = \frac{3}{4}$, so

$$DE = \frac{5}{2} \cdot \frac{3}{4} = \frac{15}{8}.$$

Thus, the correct answer is **D**.

12. Let S be a set of 6 integers taken from $\{1, 2, \dots, 12\}$ with the property that if a and b are elements of S with $a < b$, then b is not a multiple of a . What is the least possible value of an element of S ?

A 2

B 3

C 4

D 5

E 7

Solution:

Partition $\{1, \dots, 12\}$ into the six divisibility chains $\{1, 2, 4, 8\}$, $\{3, 6, 12\}$, $\{5, 10\}$, $\{7\}$, $\{9\}$, $\{11\}$. Since no element of S may divide another, at most one comes from each chain; needing 6 elements forces exactly one from each, so $7, 9, 11 \in S$.

Because $9 \in S$, $3 \notin S$, so the second chain contributes 6 or 12, and then neither 1 nor 2 can be chosen from the first chain (they divide 6 and 12). Taking 4 from the first chain works: $S = \{4, 5, 6, 7, 9, 11\}$ has the property. Hence the least possible element is 4.

Thus, the correct answer is **C**.

13. How many nonnegative integers can be written in the form

$$\begin{aligned} & a_7 \cdot 3^7 + a_6 \cdot 3^6 + a_5 \cdot 3^5 \\ & + a_4 \cdot 3^4 + a_3 \cdot 3^3 + a_2 \cdot 3^2 \\ & + a_1 \cdot 3^1 + a_0 \cdot 3^0, \end{aligned}$$

where $a_i \in \{-1, 0, 1\}$ for $0 \leq i \leq 7$?

- A 512
- B 729
- C 1094
- D 3281**
- E 59,048

Solution:

Adding 1 to every a_i gives a bijection between these expressions and the base-3 numerals for 0 through $3^8 - 1$, so exactly $3^8 = 6561$ distinct integers occur. They are symmetric about 0 (negating all a_i negates the value), so besides 0 itself, half are positive:

$$1 + \frac{1}{2}(6561 - 1) = 3281$$

nonnegative integers, namely 0 through 3280.

Thus, the correct answer is **D**.

14. The solution to the equation $\log_{3x} 4 = \log_{2x} 8$, where x is a positive real number other than $\frac{1}{3}$ or $\frac{1}{2}$, can be written as $\frac{p}{q}$, where p and q are relatively prime positive integers. What is $p + q$?

A 5

B 13

C 17

D 31

E 35

Solution:

Writing both logarithms in base 2 : $\frac{2}{\log_2 3x} = \frac{3}{\log_2 2x}$, so $2 \log_2 2x = 3 \log_2 3x$, i.e. $(2x)^2 = (3x)^3$. Then $4x^2 = 27x^3$, giving $x = \frac{4}{27}$. Since $\gcd(4, 27) = 1$, we get $p + q = 4 + 27 = 31$.

Thus, the correct answer is **D**.

15. A scanning code consists of a 7×7 grid of squares, with some of its squares colored black and the rest colored white. There must be at least one square of each color in this grid of 49 squares. A scanning code is called *symmetric* if its look does not change when the entire square is rotated by a multiple of 90° counterclockwise around its center, nor when it is reflected across a line joining opposite corners or a line joining midpoints of opposite sides. What is the total number of possible symmetric scanning codes?

- A 510
- B 1022**
- C 8190
- D 8192
- E 65,534

Solution:

Under the symmetry group of the square, the 49 cells break into orbits, and every cell in an orbit must have the same color. Give a cell coordinates (i, j) relative to the center, where $-3 \leq i, j \leq 3$. Rotations and reflections can change signs and interchange the coordinates, so each orbit has one representative with $0 \leq i \leq j \leq 3$. There are $1 + 2 + 3 + 4 = 10$ such pairs. Each orbit is black or white, giving 2^{10} colorings, but the all-black and all-white grids are excluded. So there are $2^{10} - 2 = 1022$ symmetric scanning codes.

Thus, the correct answer is **B**.

16. Which of the following describes the set of values of a for which the curves $x^2 + y^2 = a^2$ and $y = x^2 - a$ in the real xy -plane intersect at exactly 3 points?

A $a = \frac{1}{4}$

B $\frac{1}{4} < a < \frac{1}{2}$

C $a > \frac{1}{4}$

D $a = \frac{1}{2}$

E $a > \frac{1}{2}$

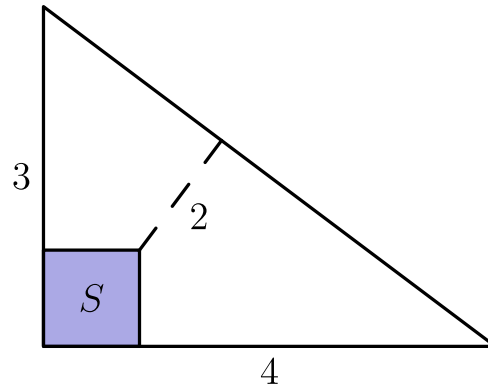
Solution:

Substituting $x^2 = y + a$ into $x^2 + y^2 = a^2$ gives $y^2 + y + (a - a^2) = 0$, which factors as $(y + 1 - a)(y + a) = 0$, so $y = a - 1$ or $y = -a$. These correspond to $x^2 = 2a - 1$ and $x^2 = 0$.

The equation $x^2 = 0$ always gives the single point $(0, -a)$, the vertex of the parabola. The equation $x^2 = 2a - 1$ gives two more points exactly when $2a - 1 > 0$, i.e. $a > \frac{1}{2}$. So there are 3 intersection points precisely when $a > \frac{1}{2}$.

Thus, the correct answer is **E**.

17. Farmer Pythagoras has a field in the shape of a right triangle. The right triangle's legs have lengths of 3 and 4 units. In the corner where those sides meet at a right angle, he leaves a small unplanted square S so that from the air it looks like the right angle symbol. The rest of the field is planted. The shortest distance from S to the hypotenuse is 2 units. What fraction of the field is planted?



- A $\frac{25}{27}$
- B $\frac{26}{27}$
- C $\frac{73}{75}$
- D $\frac{145}{147}$**
- E $\frac{74}{75}$

Solution:

Place the right angle at the origin with legs on the axes, so the vertices are $(4, 0)$, $(0, 3)$, $(0, 0)$, and the square S is $[0, s] \times [0, s]$. The hypotenuse is $3x + 4y - 12 = 0$, and the distance from its nearest corner (s, s) is

$$\frac{|3s + 4s - 12|}{\sqrt{3^2 + 4^2}} = \frac{|7s - 12|}{5} = 2.$$

This gives $s = \frac{22}{7}$ or $s = \frac{2}{7}$; only $s = \frac{2}{7}$ keeps the square inside the triangle.

The field has area $\frac{1}{2} \cdot 3 \cdot 4 = 6$ and the unplanted square has area $\left(\frac{2}{7}\right)^2 = \frac{4}{49}$. The planted fraction is

$$1 - \frac{\frac{4}{49}}{6} = 1 - \frac{2}{147} = \frac{145}{147}.$$

Thus, the correct answer is **D**.

18. Triangle ABC with $AB = 50$ and $AC = 10$ has area 120. Let D be the midpoint of $\overline{AB}$, and let E be the midpoint of $\overline{AC}$. The angle bisector of $\angle BAC$ intersects $\overline{DE}$ and $\overline{BC}$ at F and G , respectively. What is the area of quadrilateral $FDBG$?

A 60

B 65

C 70

D 75

E 80

Solution:

Since D and E are midpoints, $\triangle ADE$ has $\frac{1}{4}$ the area of $\triangle ABC$, namely 30, so trapezoid $EDBC$ has area $120 - 30 = 90$.

By the Angle Bisector Theorem, G divides BC with $BG = \frac{AB}{AB+AC} \cdot BC = \frac{5}{6}BC$, and likewise F divides DE so that $DF = \frac{5}{6}DE$. Because $FDBG$ and $EDBC$ share the same height, the area of $FDBG$ is $\frac{5}{6}$ of the area of $EDBC$: $\frac{5}{6} \cdot 90 = 75$.

Thus, the correct answer is **D**.

19. Let A be the set of positive integers that have no prime factors other than 2, 3, or 5. The infinite sum

$$\begin{aligned} & \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \\ & \quad + \frac{1}{5} + \frac{1}{6} + \frac{1}{8} + \frac{1}{9} \\ & \quad + \frac{1}{10} + \frac{1}{12} + \frac{1}{15} + \frac{1}{16} \\ & \quad + \frac{1}{18} + \frac{1}{20} + \cdots \end{aligned}$$

of the reciprocals of all the elements of A can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n$?

- ☐ A 16
- ☐ B 17
- ☒ C 19
- ☐ D 23
- ☐ E 36

Solution:

Each element of A is uniquely $2^i 3^j 5^k$ with $i, j, k \geq 0$, so summing all reciprocals factors as

$$\begin{aligned}
& \left(\sum_{i \geq 0} \frac{1}{2^i} \right) \\
& \cdot \left(\sum_{j \geq 0} \frac{1}{3^j} \right) \\
& \cdot \left(\sum_{k \geq 0} \frac{1}{5^k} \right) \\
& = \frac{1}{1 - \frac{1}{2}} \cdot \frac{1}{1 - \frac{1}{3}} \\
& \quad \cdot \frac{1}{1 - \frac{1}{5}}.
\end{aligned}$$

This equals $2 \cdot \frac{3}{2} \cdot \frac{5}{4} = \frac{15}{4}$. With $\gcd(15, 4) = 1$, $m + n = 15 + 4 = 19$.

Thus, the correct answer is **C**.

20. Triangle ABC is an isosceles right triangle with $AB = AC = 3$. Let M be the midpoint of hypotenuse $\overline{BC}$. Points I and E lie on sides $\overline{AC}$ and $\overline{AB}$, respectively, so that $AI > AE$ and $AIME$ is a cyclic quadrilateral. Given that triangle EMI has area 2, the length CI can be written as $\frac{a-\sqrt{b}}{c}$, where a , b , and c are positive integers and b is not divisible by the square of any prime. What is the value of $a + b + c$?

- A 9
- B 10
- C 11
- D 12**
- E 13

Solution:

Since $\triangle ABC$ is an isosceles right triangle, $CM = BM = \frac{3}{2}\sqrt{2}$ and the base angles at B, C are 45° . As $AIME$ is cyclic with right angle at A , angle $\angle IME = 90^\circ$. Let $x = CI$ and $y = BE$. By the Law of Cosines in $\triangle MCI$,

$$\begin{aligned} IM^2 &= x^2 + \frac{9}{2} \\ &\quad - 2 \cdot x \cdot \frac{3}{2}\sqrt{2} \cdot \cos 45^\circ \\ &= x^2 - 3x + \frac{9}{2}, \end{aligned}$$

and similarly $ME^2 = y^2 - 3y + \frac{9}{2}$.

The Pythagorean Theorem in right triangles EMI and IAE gives $IM^2 + ME^2 = (3 - x)^2 + (3 - y)^2$, which simplifies to $x + y = 3$. The area condition $\frac{1}{2}IM \cdot ME = 2$ means $IM^2 \cdot ME^2 = 16$. Substituting $y = 3 - x$ makes $ME^2 = x^2 - 3x + \frac{9}{2} = IM^2$, so $(x^2 - 3x + \frac{9}{2})^2 = 16$, hence $x^2 - 3x + \frac{9}{2} = 4$, i.e. $x^2 - 3x + \frac{1}{2} = 0$.

Since $AI > AE$ forces $y > x$, we take the smaller root $x = \frac{3-\sqrt{7}}{2}$. Then $a + b + c = 3 + 7 + 2 = 12$.

Thus, the correct answer is **D**.

21. Which of the following polynomials has the greatest real root?

A $x^{19} + 2018x^{11} + 1$

B $x^{17} + 2018x^{11} + 1$

C $x^{19} + 2018x^{13} + 1$

D $x^{17} + 2018x^{13} + 1$

E $2019x + 2018$

Solution:

Each polynomial in choices A–D has no positive root and exactly one negative root, which lies in $(-1, 0)$ (it is positive at 0 and negative at -1) and is increasing there. On the interval $(-1, 0)$, $x^{19} > x^{17}$ and $x^{13} > x^{11}$. Thus each of A, C, and D has a larger value than B at every point of this interval, so each crosses 0 to the left of B. Therefore B has the greatest root among A–D.

The linear choice E has root $e = -\frac{2018}{2019}$. Since $\frac{2018}{2019} > \frac{9}{10}$, we have $e^{17} + 2018e^{11} + 1 < -2018\left(\frac{9}{10}\right)^{11} + 1 < 0$. Because B is increasing and is negative at e , its root lies to the right of E's. Hence B has the greatest real root.

Thus, the correct answer is **B**.

22. The solutions to the equations $z^2 = 4 + 4\sqrt{15}i$ and $z^2 = 2 + 2\sqrt{3}i$, where $i = \sqrt{-1}$, form the vertices of a parallelogram in the complex plane. The area of this parallelogram can be written in the form $p\sqrt{q} - r\sqrt{s}$, where p, q, r , and s are positive integers and neither q nor s is divisible by the square of any prime number. What is $p + q + r + s$?

A 20

B 21

C 22

D 23

E 24

Solution:

Writing $z = a + bi$ with $(a + bi)^2 = 4 + 4\sqrt{15}i$ gives $a^2 - b^2 = 4$ and $2ab = 4\sqrt{15}$. Then $a^4 - 4a^2 - 60 = 0$, so $(a^2 - 10)(a^2 + 6) = 0$, yielding $a = \pm\sqrt{10}$, $b = \pm\sqrt{6}$. The vertices from the first equation are $\pm(\sqrt{10} + \sqrt{6}i)$. The same method on $z^2 = 2 + 2\sqrt{3}i$ gives $\pm(\sqrt{3} + i)$.

Applying the shoelace formula to $(\sqrt{10}, \sqrt{6}), (\sqrt{3}, 1), (-\sqrt{10}, -\sqrt{6}), (-\sqrt{3}, -1)$ gives area $6\sqrt{2} - 2\sqrt{10}$. Thus $p + q + r + s = 6 + 2 + 2 + 10 = 20$.

Thus, the correct answer is **A**.

23. In $\triangle PAT$, $\angle P = 36^\circ$, $\angle A = 56^\circ$, and $PA = 10$. Points U and G lie on sides $\overline{TP}$ and $\overline{TA}$, respectively, so that $PU = AG = 1$. Let M and N be the midpoints of segments $\overline{PA}$ and $\overline{UG}$, respectively. What is the degree measure of the acute angle formed by lines MN and PA ?

A 76

B 77

C 78

D 79

E 80

Solution:

Extend PN through N to Q with $PN = NQ$. Since N is the midpoint of UG and of PQ , the quadrilateral $UPGQ$ is a parallelogram, so $GQ \parallel PT$ and $GQ = PU = 1 = AG$. Then $\angle QGA = 180^\circ - \angle T = \angle P + \angle A = 36^\circ + 56^\circ = 92^\circ$, and the isosceles triangle QGA gives $\angle QAG = \frac{1}{2}(180^\circ - 92^\circ) = 44^\circ$.

Because M , N are midpoints, MN is a midline of $\triangle QPA$, so $MN \parallel AQ$ and

$$\begin{aligned}\angle NMP &= \angle QAP \\ &= \angle QAG + \angle GAP \\ &= 44^\circ + 56^\circ = 100^\circ.\end{aligned}$$

The acute angle between line MN and PA is therefore $180^\circ - 100^\circ = 80^\circ$.

Thus, the correct answer is **E**.

24. Alice, Bob, and Carol play a game in which each of them chooses a real number between 0 and 1. The winner of the game is the one whose number is between the numbers chosen by the other two players. Alice announces that she will choose her number uniformly at random from all the numbers between 0 and 1, and Bob announces that he will choose his number uniformly at random from all the numbers between $\frac{1}{2}$ and $\frac{2}{3}$. Armed with this information, what number should Carol choose to maximize her chance of winning?

A $\frac{1}{2}$

B $\frac{13}{24}$

C $\frac{7}{12}$

D $\frac{5}{8}$

E $\frac{2}{3}$

Solution:

If $c \leq \frac{1}{2}$, Carol beats Bob automatically, so she wins only if Alice is below c , probability $c \leq \frac{1}{2}$. If $c \geq \frac{2}{3}$, she wins with probability $1 - c \leq \frac{1}{3}$. Neither case exceeds $\frac{1}{2}$.

For $\frac{1}{2} < c < \frac{2}{3}$, the chance Bob's number exceeds c is $\frac{\frac{2}{3}-c}{\frac{2}{3}-\frac{1}{2}} = 4 - 6c$, so the probability Carol is above Alice and below Bob is $c(4 - 6c)$; the reverse ordering has probability $(1 - c)(6c - 3)$. Adding,

$$\begin{aligned} &c(4 - 6c) + (1 - c)(6c - 3) \\ &= -12c^2 + 13c - 3. \end{aligned}$$

This downward parabola is maximized at $c = \frac{13}{24}$, which lies in $(\frac{1}{2}, \frac{2}{3})$, and its value exceeds $\frac{1}{2}$.

Thus, the correct answer is **B**.

25. For a positive integer n and nonzero digits a , b , and c , let A_n be the n -digit integer each of whose digits is equal to a ; let B_n be the n -digit integer each of whose digits is equal to b ; and let C_n be the $2n$ -digit (not n -digit) integer each of whose digits is equal to c . What is the greatest possible value of $a + b + c$ for which there are at least two values of n such that $C_n - B_n = A_n^2$?

A 12

B 14

C 16

D 18

E 20

Solution:

Using $A_n = a \cdot \frac{10^n - 1}{9}$, $B_n = b \cdot \frac{10^n - 1}{9}$, and $C_n = c \cdot \frac{10^{2n} - 1}{9}$, the equation $C_n - B_n = A_n^2$ becomes, after dividing by $10^n - 1$ and clearing fractions,

$$(9c - a^2) \cdot 10^n = 9b - 9c - a^2.$$

For this to hold at two different n , the coefficient of 10^n must be zero, so $9c = a^2$ and hence $9b - 9c - a^2 = 0$.

Then $c = \frac{a^2}{9}$ and $b = 2c$. So $a \in \{3, 6, 9\}$ with $c \in \{1, 4, 9\}$ and $b \in \{2, 8, 18\}$; the case $b = 18$ is not a digit. The valid triples are $(a, b, c) = (3, 2, 1)$ and $(6, 8, 4)$, and indeed $4444 - 88 = 4356 = 66^2$. The greater digit sum is $6 + 8 + 4 = 18$.

Thus, the correct answer is **D**.

Problems: <https://live.poshenloh.com/past-contests/amc12/2018A>



2018 AMC 12B Solutions

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1. Kate bakes a 20-inch by 18-inch pan of cornbread. The cornbread is cut into pieces that measure 2 inches by 2 inches. How many pieces of cornbread does the pan contain?

☒ A 90

☐ B 100

☐ C 180

☐ D 200

☐ E 360

Solution:

The pan has area $20 \cdot 18 = 360$ square inches, and each piece has area $2 \cdot 2 = 4$ square inches.

The number of pieces is

$$\frac{360}{4} = 90.$$

Thus, the correct answer is **A**.

2. Sam drove 96 miles in 90 minutes. His average speed during the first 30 minutes was 60 mph (miles per hour), and his average speed during the second 30 minutes was 65 mph. What was his average speed, in mph, during the last 30 minutes?

A 64

B 65

C 66

D 67

E 68

Solution:

In the first 30 minutes Sam covered $60 \cdot \frac{1}{2} = 30$ miles, and in the second he covered $65 \cdot \frac{1}{2} = 32.5$ miles.

The last 30 minutes covered $96 - 30 - 32.5 = 33.5$ miles, so the speed was

$$\frac{33.5}{\frac{1}{2}} = 67 \text{ mph.}$$

Thus, the correct answer is **D**.

3. A line with slope 2 intersects a line with slope 6 at the point $(40, 30)$. What is the distance between the x -intercepts of these two lines?

A 5

B 10

C 20

D 25

E 50

Solution:

The line of slope 2 is $y - 30 = 2(x - 40)$; setting $y = 0$ gives $x = 25$. The line of slope 6 is $y - 30 = 6(x - 40)$; setting $y = 0$ gives $x = 35$.

The distance between the intercepts is $|35 - 25| = 10$.

Thus, the correct answer is **B**.

4. A circle has a chord of length 10, and the distance from the center of the circle to the chord is 5. What is the area of the circle?

A 25π

B 50π

C 75π

D 100π

E 125π

Solution:

Dropping a perpendicular from the center to the chord bisects it, forming a right triangle with legs 5 (half the chord) and 5 (the distance), and hypotenuse r .

Then $r^2 = 5^2 + 5^2 = 50$, so the area is $\pi r^2 = 50\pi$.

Thus, the correct answer is **B**.

5. How many subsets of $\{2, 3, 4, 5, 6, 7, 8, 9\}$ contain at least one prime number?

A 128

B 192

C 224

D 240

E 256

Solution:

The set has 8 elements, giving $2^8 = 256$ subsets. The subsets with no prime use only the four non-primes $\{4, 6, 8, 9\}$, and there are $2^4 = 16$ of these.

So the number containing at least one prime is $256 - 16 = 240$.

Thus, the correct answer is **D**.

6. Suppose S cans of soda can be purchased from a vending machine for Q quarters. Which of the following expressions describes the number of cans of soda that can be purchased for D dollars, where 1 dollar is worth 4 quarters?

A $\frac{4DQ}{S}$

B $\frac{4DS}{Q}$

C $\frac{4Q}{DS}$

D $\frac{DQ}{4S}$

E $\frac{DS}{4Q}$

Solution:

One can costs $\frac{Q}{S}$ quarters, which is $\frac{Q}{4S}$ dollars. The number of cans that D dollars can buy is

$$\frac{D}{\frac{Q}{4S}} = \frac{4DS}{Q}.$$

Thus, the correct answer is **B**.

7. What is the value of

$$\log_3 7 \cdot \log_5 9 \cdot \log_7 11 \\ \cdot \log_9 13 \cdots \log_{21} 25 \cdot \log_{23} 27?$$

A 3

B $3 \log_7 23$

C 6

D 9

E 10

Solution:

The factors split into two telescoping chains. The odd-position factors form

$$\log_3 7 \cdot \log_7 11 \\ \cdot \log_{11} 13 \cdots \log_{23} 27 \\ = \log_3 27 = 3,$$

and the even-position factors form

$$\log_5 9 \cdot \log_9 13 \cdots \log_{21} 25 \\ = \log_5 25 = 2.$$

The product is $3 \cdot 2 = 6$.

Thus, the correct answer is **C**.

8. Line segment $\overline{AB}$ is a diameter of a circle with $AB = 24$. Point C , not equal to A or B , lies on the circle. As point C moves around the circle, the centroid (center of mass) of $\triangle ABC$ traces out a closed curve missing two points. To the nearest positive integer, what is the area of the region bounded by this curve?

A 25

B 38

C 50

D 63

E 75

Solution:

Let O be the center of the circle. The centroid of $\triangle ABC$ is the average of A , B , and C ; since O is the midpoint of $\overline{AB}$, the centroid lies one-third of the way from O to C .

As C traces the circle of radius 12, the centroid traces a circle of radius $\frac{1}{3} \cdot 12 = 4$. Its area is $16\pi \approx 50$.

Thus, the correct answer is **C**.

9. What is

$$\sum_{i=1}^{100} \sum_{j=1}^{100} (i + j)?$$

A 100,100

B 500,500

C 505,000

D 1,001,000

E 1,010,000

Solution:

Splitting the sum,

$$\begin{aligned} & \sum_{i=1}^{100} \sum_{j=1}^{100} (i + j) \\ &= \sum_{i=1}^{100} \sum_{j=1}^{100} i \\ &+ \sum_{i=1}^{100} \sum_{j=1}^{100} j \\ &= 100 \sum_{i=1}^{100} i \\ &+ 100 \sum_{j=1}^{100} j. \end{aligned}$$

Since $\sum_{k=1}^{100} k = 5050$, this equals $100 \cdot 5050 + 100 \cdot 5050 = 1,010,000$.

Thus, the correct answer is **E**.

10. A list of 2018 positive integers has a unique mode, which occurs exactly 10 times. What is the least number of distinct values that can occur in the list?

- A 202
- B 223
- C 224
- D 225**
- E 234

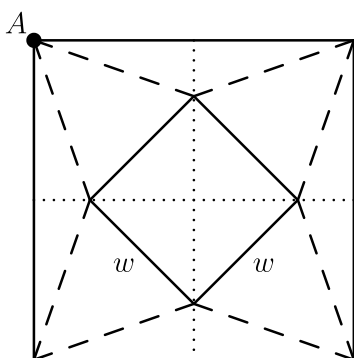
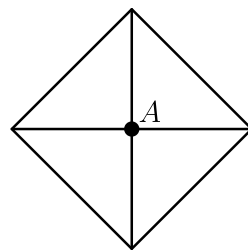
Solution:

The mode uses 10 of the entries, leaving 2008. Because the mode is unique, every other value appears at most 9 times, so at least $\lceil \frac{2008}{9} \rceil = 224$ distinct non-mode values are needed.

Adding the mode gives $224 + 1 = 225$. This is achievable: use 9 copies each of 1 through 223, ten copies of 224, and one copy of 225.

Thus, the correct answer is **D**.

11. A closed box with a square base is to be wrapped with a square sheet of wrapping paper. The box is centered on the wrapping paper with the vertices of the base lying on the midlines of the square sheet of paper, as shown in the figure on the left. The four corners of the wrapping paper are to be folded up over the sides and brought together to meet at the center of the top of the box, point A in the figure on the right. The box has base length w and height h . What is the area of the sheet of wrapping paper?



- A $2(w + h)^2$
- B $\frac{(w + h)^2}{2}$
- C $2w^2 + 4wh$
- D $2w^2$
- E w^2h

Solution:

Following a fold from a corner of the paper to the center of the box top, the distance from a corner of the sheet to its center is

$$\frac{w}{2} + h + \frac{w}{2} = w + h.$$

That segment is a leg of a $45\text{-}45\text{-}90$ triangle whose hypotenuse is a full side of the square sheet, so the side length is $\sqrt{2}(w + h)$.

The area of the sheet is $(\sqrt{2}(w + h))^2 = 2(w + h)^2$.

Thus, the correct answer is **A**.

12. Side $\overline{AB}$ of $\triangle ABC$ has length 10. The bisector of angle A meets $\overline{BC}$ at D , and $CD = 3$. The set of all possible values of AC is an open interval (m, n) . What is $m + n$?

A 16

B 17

C 18

D 19

E 20

Solution:

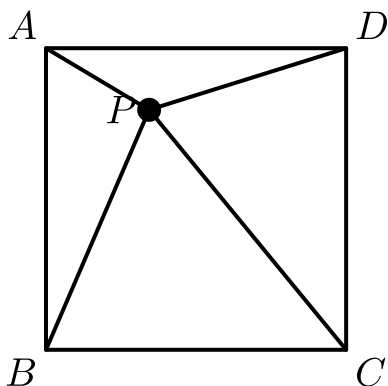
Let $q = AC$ and $r = BD$. The angle bisector theorem gives $\frac{q}{3} = \frac{10}{r}$, so $r = \frac{30}{q}$.

Applying the triangle inequalities to sides q , 10, and $3 + r$ and substituting $r = \frac{30}{q}$ yields $(q - 15)(q + 2) < 0$ and $(q - 3)(q + 10) > 0$ (the third inequality holds automatically). Together these force $3 < q < 15$.

So $(m, n) = (3, 15)$ and $m + n = 18$.

Thus, the correct answer is **C**.

13. Square $ABCD$ has side length 30. Point P lies inside the square so that $AP = 12$ and $BP = 26$. The centroids of $\triangle ABP$, $\triangle BCP$, $\triangle CDP$, and $\triangle DAP$ are the vertices of a convex quadrilateral. What is the area of that quadrilateral?



- ☐ A $100\sqrt{2}$
- ☐ B $100\sqrt{3}$
- ☒ C 200
- ☐ D $200\sqrt{2}$
- ☐ E $200\sqrt{3}$

Solution:

Place $A = (0, 30)$, $B = (0, 0)$, $C = (30, 0)$, $D = (30, 30)$, and $P = (3x, 3y)$. Averaging the vertices, the four centroids are

$$\begin{aligned} &(x, y + 10), \\ &(x + 10, y), \\ &(x + 20, y + 10), \\ &(x + 10, y + 20). \end{aligned}$$

These form a square whose diagonals, one horizontal and one vertical, each have length 20. Its area is $\frac{1}{2} \cdot 20 \cdot 20 = 200$, independent of where P lies.

Thus, the correct answer is **C**.

14. Joey and Chloe and their daughter Zoe all have the same birthday. Joey is 1 year older than Chloe, and Zoe is exactly 1 year old today. Today is the first of the 9 birthdays on which Chloe's age will be an integral multiple of Zoe's age. What will be the sum of the two digits of Joey's age the next time his age is a multiple of Zoe's age?

- A 7
- B 8
- C 9
- D 10
- E 11**

Solution:

Let Chloe be n today, so she is $n - 1$ years older than Zoe. In y years Chloe's age $n + y$ is a multiple of Zoe's age $1 + y$ exactly when $1 + y$ divides $n - 1$. Having 9 such birthdays means $n - 1$ has exactly 9 divisors.

A number with exactly 9 divisors has the form $p^2 q^2$ for distinct primes p, q , or p^8 .

Because Joey's age at the requested future birthday has two digits, $n - 1 < 99$; the only possibility is $2^2 \cdot 3^2 = 36$. So Chloe is 37 and Joey is 38.

Joey's age $38 + y$ is a multiple of $1 + y$ exactly when $1 + y$ divides 37. The next time is $y = 36$, making Joey 74, with digit sum $7 + 4 = 11$.

Thus, the correct answer is **E**.

15. How many 3-digit positive odd multiples of 3 do not include the digit 3?

A 96

B 97

C 98

D 102

E 120

Solution:

Write the number as $\overline{abc}$. The hundreds digit a has 8 choices (1, 2, 4, 5, 6, 7, 8, 9), and the units digit c has 4 choices (1, 5, 7, 9).

The tens digit b may be any of $\{0, 1, 2, 4, 5, 6, 7, 8, 9\}$. These split into three residue classes mod 3 of equal size $\{0, 6, 9\}$, $\{1, 4, 7\}$, $\{2, 5, 8\}$, so exactly 3 choices of b make $a + b + c$ divisible by 3.

The count is $8 \cdot 4 \cdot 3 = 96$.

Thus, the correct answer is **A**.

16. The solutions to the equation $(z + 6)^8 = 81$ are connected in the complex plane to form a convex regular polygon, three of whose vertices are labeled A , B , and C . What is the least possible area of $\triangle ABC$?

- A $\frac{1}{6}\sqrt{6}$
- B $\frac{3}{2}\sqrt{2} - \frac{3}{2}$
- C $2\sqrt{3} - 2\sqrt{2}$
- D $\frac{1}{2}\sqrt{2}$
- E $\sqrt{3} - 1$

Solution:

Translating by 6, the solutions of $z^8 = 81$ are eight points on a circle of radius $81^{\frac{1}{8}} = \sqrt{3}$, forming a regular octagon. The minimum-area triangle uses three consecutive vertices.

Take $A = (\frac{1}{2}\sqrt{6}, \frac{1}{2}\sqrt{6})$, $B = (\sqrt{3}, 0)$, and $C = (\frac{1}{2}\sqrt{6}, -\frac{1}{2}\sqrt{6})$. Then $AC = \sqrt{6}$ and the height is $\sqrt{3} - \frac{1}{2}\sqrt{6}$, so the area is

$$\begin{aligned} \frac{1}{2} \cdot \sqrt{6} \left(\sqrt{3} - \frac{1}{2}\sqrt{6} \right) \\ = \frac{3}{2}\sqrt{2} - \frac{3}{2}. \end{aligned}$$

Thus, the correct answer is **B**.

17. Let p and q be positive integers such that

$$\frac{5}{9} < \frac{p}{q} < \frac{4}{7}$$

and q is as small as possible. What is $q - p$?

A 7

B 11

C 13

D 17

E 19

Solution:

From $\frac{5}{9} < \frac{p}{q}$ we get $9p - 5q \geq 1$, and from $\frac{p}{q} < \frac{4}{7}$ we get $4q - 7p \geq 1$. Now

$$\begin{aligned} \frac{1}{63} &= \frac{4}{7} - \frac{5}{9} \\ &= \frac{4q - 7p}{7q} + \frac{9p - 5q}{9q} \\ &\geq \frac{1}{7q} + \frac{1}{9q} \\ &= \frac{16}{63q}. \end{aligned}$$

Hence $q \geq 16$. With $q = 16$, the fraction $\frac{9}{16}$ lies strictly between $\frac{5}{9}$ and $\frac{4}{7}$, so $p = 9$ and $q - p = 16 - 9 = 7$.

Thus, the correct answer is **A**.

18. A function f is defined recursively by $f(1) = f(2) = 1$ and

$$f(n) = f(n-1) - f(n-2) + n$$

for all integers $n \geq 3$. What is $f(2018)$?

A 2016

B 2017

C 2018

D 2019

E 2020

Solution:

Repeatedly substituting the recursion into itself gives

$$f(n) = f(n-6) + 6.$$

So f increases by 6 every time n increases by 6.

Since $2018 = 2 + 6 \cdot 336$, we have $f(2018) = f(2) + 6 \cdot 336 = 1 + 2016 = 2017$.

Thus, the correct answer is **B**.

19. Mary chose an even 4-digit number n . She wrote down all the divisors of n in increasing order from left to right: $1, 2, \dots, \frac{n}{2}, n$. At some moment Mary wrote **323** as a divisor of n . What is the smallest possible value of the next divisor written to the right of **323**?

A 324

B 330

C 340

D 361

E 646

Solution:

Let d be the next divisor after **323**. If $\gcd(d, 323) = 1$, then $323d$ divides n , forcing $n \geq 323d > 323^2 > 9999$, impossible for a 4-digit number. So d shares a prime factor with $323 = 17 \cdot 19$.

Then $d - 323 \geq \gcd(d, 323) \geq 17$, so $d \geq 340$. Indeed $d = 340 = 17 \cdot 20$ occurs for $n = 17 \cdot 19 \cdot 20 = 6460$, which is even and 4-digit.

Thus, the correct answer is **C**.

20. Let $ABCDEF$ be a regular hexagon with side length 1. Denote by X, Y , and Z the midpoints of sides AB, CD , and EF , respectively. What is the area of the convex hexagon whose interior is the intersection of the interiors of $\triangle ACE$ and $\triangle XYZ$?

- A $\frac{3}{8}\sqrt{3}$
- B $\frac{7}{16}\sqrt{3}$
- C $\frac{15}{32}\sqrt{3}$**
- D $\frac{1}{2}\sqrt{3}$
- E $\frac{9}{16}\sqrt{3}$

Solution:

Place the regular hexagon on the unit circle with $A = (1, 0)$, $C = (-\frac{1}{2}, \frac{\sqrt{3}}{2})$, and $E = (-\frac{1}{2}, -\frac{\sqrt{3}}{2})$. The three specified midpoints are $X = (\frac{3}{4}, \frac{\sqrt{3}}{4})$, $Y = (-\frac{3}{4}, \frac{\sqrt{3}}{4})$, and $Z = (0, -\frac{\sqrt{3}}{2})$.

Intersecting the side lines of $\triangle ACE$ and $\triangle XYZ$ gives the six vertices of their common interior, in cyclic order:

$$\begin{aligned} &(-\frac{1}{2}, 0), \quad (-\frac{1}{8}, -\frac{3\sqrt{3}}{8}), \\ &(\frac{1}{4}, -\frac{\sqrt{3}}{4}), \quad (\frac{5}{8}, \frac{\sqrt{3}}{8}), \\ &(\frac{1}{4}, \frac{\sqrt{3}}{4}), \quad (-\frac{1}{2}, \frac{\sqrt{3}}{4}). \end{aligned}$$

The shoelace formula applied to these vertices gives area $\frac{15\sqrt{3}}{32}$.

Thus, the correct answer is **C**.

21. In $\triangle ABC$ with side lengths $AB = 13$, $AC = 12$, and $BC = 5$, let O and I denote the circumcenter and incenter, respectively. A circle with center M is tangent to the legs AC and BC and to the circumcircle of $\triangle ABC$. What is the area of $\triangle MOI$?

- A $\frac{5}{2}$
- B $\frac{11}{4}$
- C 3
- D $\frac{13}{4}$
- E $\frac{7}{2}$**

Solution:

Since $5^2 + 12^2 = 13^2$, the triangle is right-angled at C . Set $C = (0, 0)$, $A = (12, 0)$, and $B = (0, 5)$. Then O is the midpoint of $\overline{AB}$, namely $O = (6, \frac{5}{2})$, with circumradius $\frac{13}{2}$. The inradius is $\frac{\text{area}}{s} = \frac{30}{15} = 2$, so $I = (2, 2)$.

Because M 's circle is tangent to both legs, $M = (\rho, \rho)$. Internal tangency to the circumcircle gives $MO = \frac{13}{2} - \rho$. Setting this equal to $\sqrt{(\rho - 6)^2 + (\rho - \frac{5}{2})^2}$ and solving gives $\rho = 4$, so $M = (4, 4)$.

The shoelace formula on $M = (4, 4)$, $O = (6, \frac{5}{2})$, $I = (2, 2)$ gives area $\frac{7}{2}$.

Thus, the correct answer is **E**.

22. Consider polynomials $P(x)$ of degree at most 3, each of whose coefficients is an element of $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. How many such polynomials satisfy $P(-1) = -9$?

A 110

B 143

C 165

D 220

E 286

Solution:

Write $P(x) = ax^3 + bx^2 + cx + d$ with each of a, b, c, d in $\{0, \dots, 9\}$. The condition is $-a + b - c + d = -9$.

Let $a' = 9 - a$ and $c' = 9 - c$, both in $[0, 9]$. Then $a' + b + c' + d = 9$. By stars and bars the number of nonnegative solutions is $\binom{9+3}{3} = \binom{12}{3} = 220$, and each automatically satisfies the upper bounds since the sum is 9.

Thus, the correct answer is **D**.

23. Ajay is standing at point A near Pontianak, Indonesia, 0° latitude and 110° E longitude. Billy is standing at point B near Big Baldy Mountain, Idaho, USA, 45° N latitude and 115° W longitude. Assume that Earth is a perfect sphere with center C . What is the degree measure of $\angle ACB$?

A 105

B $112\frac{1}{2}$

C 120

D 135

E 150

Solution:

The longitudes differ by $360^\circ - (110^\circ + 115^\circ) = 135^\circ$, and B is at latitude 45° N. Place $A = (1, 0, 0)$ on the unit sphere.

Then $B = (\cos 45^\circ \cos 135^\circ, \cos 45^\circ \sin 135^\circ, \sin 45^\circ) = \left(-\frac{1}{2}, \frac{1}{2}, \frac{\sqrt{2}}{2}\right)$. The dot product is $A \cdot B = -\frac{1}{2}$, so $\cos \angle ACB = -\frac{1}{2}$ and $\angle ACB = 120^\circ$.

Thus, the correct answer is **C**.

24. Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x . How many real numbers x satisfy the equation $x^2 + 10,000\lfloor x \rfloor = 10,000x$?

A 197

B 198

C 199

D 200

E 201

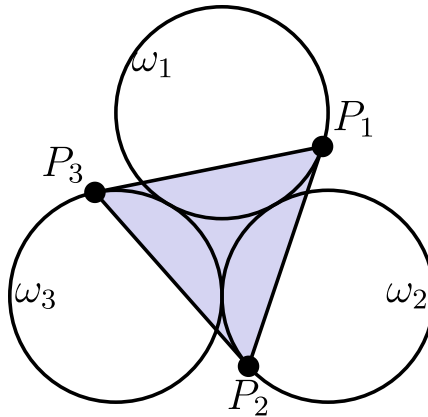
Solution:

Let $\{x\} = x - \lfloor x \rfloor$. The equation becomes $x^2 = 10,000\{x\}$, so $\frac{x^2}{10,000} = \{x\}$. Since $0 \leq \{x\} < 1$, we need $0 \leq x^2 < 10,000$, i.e. $-100 < x < 100$.

On each interval $[k, k + 1)$, write $x = k + t$ with $0 \leq t < 1$. The equation becomes $(k + t)^2 - 10,000t = 0$. For $-100 \leq k \leq 98$, the left side is strictly decreasing; at $t = 0$ it is $k^2 \geq 0$, while as t approaches 1 it approaches $(k + 1)^2 - 10,000 < 0$. Thus each of these intervals contains exactly one solution. There are $98 - (-100) + 1 = 199$ such intervals.

Thus, the correct answer is **C**.

25. Circles ω_1, ω_2 , and ω_3 each have radius 4 and are placed in the plane so that each circle is externally tangent to the other two. Points P_1, P_2 , and P_3 lie on ω_1, ω_2 , and ω_3 , respectively, so that $P_1P_2 = P_2P_3 = P_3P_1$ and line P_iP_{i+1} is tangent to ω_i for each $i = 1, 2, 3$, where $P_4 = P_1$. See the figure below. The area of $\triangle P_1P_2P_3$ can be written in the form $\sqrt{a} + \sqrt{b}$, where a and b are positive integers. What is $a + b$?



A 546

B 548

C 550

D 552

E 554

Solution:

Let O_i be the center of ω_i , and let K be the intersection of lines O_1P_1 and O_2P_2 . Because $\angle P_1P_2P_3 = 60^\circ$, triangle P_2KP_1 is a 30° - 60° - 90° triangle. With $d = P_1K$, we get $P_2K = 2d$ and $P_1P_2 = \sqrt{3}d$.

The Law of Cosines in $\triangle O_1KO_2$ (with $O_1O_2 = 8$) gives

$$\begin{aligned} 8^2 &= (d + 4)^2 + (2d - 4)^2 \\ &\quad - 2(d + 4)(2d - 4) \cos 60^\circ, \end{aligned}$$

which simplifies to $3d^2 - 12d - 16 = 0$, so $d = 2 + \frac{2}{3}\sqrt{21}$.

Then $P_1P_2 = \sqrt{3}d = 2\sqrt{3} + 2\sqrt{7}$, and the area is

$$\begin{aligned}\frac{\sqrt{3}}{4} (2\sqrt{3} + 2\sqrt{7})^2 \\ &= 10\sqrt{3} + 6\sqrt{7} \\ &= \sqrt{300} + \sqrt{252}.\end{aligned}$$

So $a + b = 300 + 252 = 552$.

Thus, the correct answer is **D**.

Problems: <https://live.poshenloh.com/past-contests/amc12/2018B>

