AMC 12 SOLUTION ARCHIVE

2017

12A and 12B full explanations

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2017 AMC 12A Solutions

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1. Pablo buys popsicles for his friends. The store sells single popsicles for $\$1$ each, 3-popsicle boxes for $\$2$, and 5-popsicle boxes for $\$3$. What is the greatest number of popsicles that Pablo can buy with $\$8$?

A 8

B 11

C 12

D 13

E 15

Solution:

The cheapest popsicles come from the 5-popsicle box, at $\frac{\$3}{5} = \$0.60$ each. Even at that rate, 14 popsicles would cost $14 \cdot \$0.60 = \$8.40$, more than $\$8$.

So Pablo can buy at most 13, and he achieves this with two 5-boxes for $\$6$ and one 3-box for $\$2$, giving $2 \cdot 5 + 3 = 13$ popsicles.

Thus, the correct answer is **D**.

2. The sum of two nonzero real numbers is 4 times their product. What is the sum of the reciprocals of the two numbers?

A 1

B 2

C 4

D 8

E 12

Solution:

Let the numbers be x and y, so $x + y = 4xy$.

Dividing both sides by xy gives

$$\frac{x+y}{xy} = 4,$$

and the left side is exactly $\frac{1}{y} + \frac{1}{x}$. So the sum of the reciprocals is 4.

Thus, the correct answer is **C**.

3. Ms. Carroll promised that anyone who got all the multiple choice questions right on the upcoming exam would receive an A on the exam. Which one of these statements necessarily follows logically?

A. If Lewis did not receive an A, then he got all of the multiple choice questions wrong.

B. If Lewis did not receive an A, then he got at least one of the multiple choice questions wrong.

C. If Lewis got at least one of the multiple choice questions wrong, then he did not receive an A.

D. If Lewis received an A, then he got all of the multiple choice questions right.

E. If Lewis received an A, then he got at least one of the multiple choice questions right.

Solution:

The promise is "all right $\Rightarrow$ A." An implication is equivalent only to its contrapositive: "not A $\Rightarrow$ not all right."

"Not all right" means at least one question was wrong, which is exactly statement B. The converse and inverse do not follow, and getting "all wrong" is a much stronger claim than the negation.

Thus, the correct answer is **B**.

4. Jerry and Silvia wanted to go from the southwest corner of a square field to the northeast corner. Jerry walked due east and then due north to reach the goal, but Silvia headed northeast and reached the goal walking in a straight line. Which of the following is closest to how much shorter Silvia's trip was, compared to Jerry's trip?

A 30%

B 40%

C 50%

D 60%

E 70%

Solution:

If the square has side x, Jerry walks $x + x = 2x$, while Silvia walks the diagonal $\sqrt{x^2 + x^2} = x\sqrt{2}$.

The fraction by which Silvia's trip is shorter is

$$\frac{2x - x\sqrt{2}}{2x} = 1 - \frac{\sqrt{2}}{2}$$
$$\approx 1 - 0.707 = 0.293.$$

This is closest to 30%.

Thus, the correct answer is **A**.

5. At a gathering of 30 people, there are 20 people who all know each other and 10 people who know no one. People who know each other hug, and people who do not know each other shake hands. How many handshakes occur?

A 240

B 245

C 290

D 480

E 490

Solution:

Each of the 20 people who know each other shakes hands with only the 10 strangers. Each of the 10 strangers shakes hands with all 29 other people.

Summing handshake counts and dividing by 2 (each handshake involves two people) gives

$$\begin{aligned} &\frac{1}{2}(20 \cdot 10 + 10 \cdot 29) \\ &= \frac{1}{2}(200 + 290) = 245. \end{aligned}$$

Thus, the correct answer is **B**.

6. Joy has 30 thin rods, one each of every integer length from 1 cm through 30 cm. She places the rods with lengths 3 cm, 7 cm, and 15 cm on a table. She then wants to choose a fourth rod that she can put with these three to form a quadrilateral with positive area. How many of the remaining rods can she choose as the fourth rod?

A 16

B 17

C 18

D 19

E 20

Solution:

Four lengths form a quadrilateral with positive area if and only if the longest is strictly less than the sum of the other three. With a fourth rod of length n, this requires $15 < 3 + 7 + n$ and $n < 3 + 7 + 15$, so

$$5 < n < 25.$$

The integers from 6 to 24 give 19 values, but the rods of length 7 and 15 are already on the table, leaving $19 - 2 = 17$ choices.

Thus, the correct answer is **B**.

7. Define a function on the positive integers recursively by $f(1) = 2, f(n) = f(n-1) + 1$ if n is even, and $f(n) = f(n-2) + 2$ if n is odd and greater than 1. What is $f(2017)$?

A 2017

B 2018

C 4034

D 4035

E 4036

Solution:

Listing values: $f(1) = 2, f(2) = f(1) + 1 = 3, f(3) = f(1) + 2 = 4, f(4) = f(3) + 1 = 5$, suggesting $f(n) = n + 1$.

Both rules are consistent with $f(n) = n + 1$: for even n, $(n-1) + 1 + 1 = n + 1$, and for odd n, $(n-2) + 1 + 2 = n + 1$. Since the recursion determines f uniquely, $f(2017) = 2018$.

Thus, the correct answer is **B**.

8. The region consisting of all points in three-dimensional space within 3 units of line segment AB has volume 216π. What is the length AB?

A 6

B 12

C 18

D 20

E 24

Solution:

Let $h = AB$. The region is a cylinder of radius 3 and height h with a hemisphere of radius 3 on each end.

The cylinder has volume $\pi \cdot 3^2 \cdot h = 9\pi h$, and the two hemispheres together form a sphere of volume $\frac{4}{3}\pi \cdot 3^3 = 36\pi$. So

$$9\pi h + 36\pi = 216\pi,$$

giving $h = 20$.

Thus, the correct answer is **D**.

9. Let S be the set of points (x, y) in the coordinate plane such that two of the three quantities $3, x+2$, and $y-4$ are equal and the third of the three quantities is no greater than this common value. Which of the following is a correct description of S?

A a single point

B two intersecting lines

C three lines whose pairwise intersections are three distinct points

D a triangle

E three rays with a common endpoint

Solution:

Consider which two of $3, x+2, y-4$ are the (equal) larger pair.

If $3 = x+2 \geq y-4$: then $x = 1$ and $y \leq 7$, a downward ray from $(1, 7)$. If $3 = y-4 \geq x+2$: then $y = 7$ and $x \leq 1$, a leftward ray from $(1, 7)$. If $x+2 = y-4 \geq 3$: then $y = x+6$ and $x \geq 1$, a ray from $(1, 7)$ going up and to the right.

All three rays share the endpoint $(1, 7)$, so S is three rays with a common endpoint.

Thus, the correct answer is **E**.

10. Chloé chooses a real number uniformly at random from the interval $[0, 2017]$. Independently, Laurent chooses a real number uniformly at random from the interval $[0, 4034]$. What is the probability that Laurent's number is greater than Chloé's number?

A $\frac{1}{2}$

B $\frac{2}{3}$

C $\frac{3}{4}$

D $\frac{5}{6}$

E $\frac{7}{8}$

Solution:

With probability $\frac{1}{2}$, Laurent's number lies in $[2017, 4034]$, which exceeds any number Chloé could choose, so he wins for certain.

With the other probability $\frac{1}{2}$, Laurent's number lies in $[0, 2017]$, matching Chloé's interval; by symmetry he is larger half the time. The total probability is

$$\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4}.$$

Thus, the correct answer is **C**.

11. Claire adds the degree measures of the interior angles of a convex polygon and arrives at a sum of 2017. She then discovers that she forgot to include one angle. What is the degree measure of the forgotten angle?

A 37

B 63

C 117

D 143

E 163

Solution:

If the polygon has n sides and the forgotten angle is α, then $(n-2)180 = 2017 + \alpha$. Since $0 < \alpha < 180$,

$$2017 < (n-2)180 < 2197.$$

The only multiple of 180 in this range is $2160 = (14-2)180$, so $n = 14$ and

$$\alpha = 2160 - 2017 = 143.$$

Thus, the correct answer is **D**.

12. There are 10 horses, named Horse 1, Horse 2, . . . , Horse 10. They get their names from how many minutes it takes them to run one lap around a circular race track: Horse k runs one lap in exactly k minutes. At time 0 all the horses are together at the starting point on the track. The horses start running in the same direction, and they keep running around the circular track at their constant speeds. The least time $S > 0$, in minutes, at which all 10 horses will again simultaneously be at the starting point is $S = 2520$. Let $T > 0$ be the least time, in minutes, such that at least 5 of the horses are again at the starting point. What is the sum of the digits of T?

A 2

B 3

C 4

D 5

E 6

Solution:

Horse k is at the starting point at time t precisely when $k \mid t$. So we want the smallest t with at least 5 divisors among $1, 2, \ldots, 10$.

The positive integers below 12 have at most 4 divisors, while 12 is divisible by $1, 2, 3, 4$, and 6. Thus $T = 12$, and the sum of its digits is $1 + 2 = 3$.

Thus, the correct answer is **B**.

13. Driving at a constant speed, Sharon usually takes 180 minutes to drive from her house to her mother's house. One day Sharon begins the drive at her usual speed, but after driving $\frac{1}{3}$ of the way, she hits a bad snowstorm and reduces her speed by 20 miles per hour. This time the trip takes her a total of 276 minutes. How many miles is the drive from Sharon's house to her mother's house?

A 132

B 135

C 138

D 141

E 144

Solution:

Let the distance be d miles and the usual speed r mph. Since the usual trip is 3 hours, $d = 3r$.

The first $\frac{1}{3}$ of the drive takes $\frac{1}{3} \cdot 180 = 60$ minutes at speed r, so the remaining $\frac{2}{3}$ takes $276 - 60 = 216$ minutes $= \frac{18}{5}$ hours at speed $r - 20$.

That final portion covers $\frac{2}{3}d = 2r$ miles, so

$$2r = (r - 20) \cdot \frac{18}{5}.$$

Solving gives $10r = 18r - 360$, so $r = 45$ and $d = 3 \cdot 45 = 135$.

Thus, the correct answer is **B**.

14. Alice refuses to sit next to either Bob or Carla. Derek refuses to sit next to Eric. How many ways are there for the five of them to sit in a row of 5 chairs under these conditions?

A 12

B 16

C 28

D 32

E 40

Solution:

Let X, Y, Z be the seatings where Alice-Bob, Alice-Carla, and Derek-Eric are adjacent, respectively. The answer is $5! - |X \cup Y \cup Z|$.

Treating a forbidden pair as a block gives $|X| = |Y| = |Z| = 2 \cdot 4! = 48$. For intersections, $|X \cap Y| = 2 \cdot 3! = 12$ (Alice between Bob and Carla), $|X \cap Z| = |Y \cap Z| = 2 \cdot 2 \cdot 3! = 24$, and $|X \cap Y \cap Z| = 2 \cdot 2 \cdot 2! = 8$.

By inclusion-exclusion, $|X \cup Y \cup Z| = (48 \cdot 3) - (12 + 24 + 24) + 8 = 92$, so the answer is $120 - 92 = 28$.

Thus, the correct answer is **C**.

15. Let $f(x) = \sin x + 2\cos x + 3\tan x$, using radian measure for the variable x. In what interval does the smallest positive value of x for which $f(x) = 0$ lie?

A $(0, 1)$

B $(1, 2)$

C $(2, 3)$

D $(3, 4)$

E $(4, 5)$

Solution:

For $0 < x < \dfrac{\pi}{2}$ all three terms are positive, so $f(x) > 0$. For $\dfrac{\pi}{2} < x < \pi$, set $s = \sin x > 0$ and $t = -\cos x > 0$. Since $0 < t \leq 1$,

$$\begin{aligned} f(x) &= s - 2t - \frac{3s}{t} \\ &= -2t - s\left(\frac{3}{t} - 1\right) < 0. \end{aligned}$$

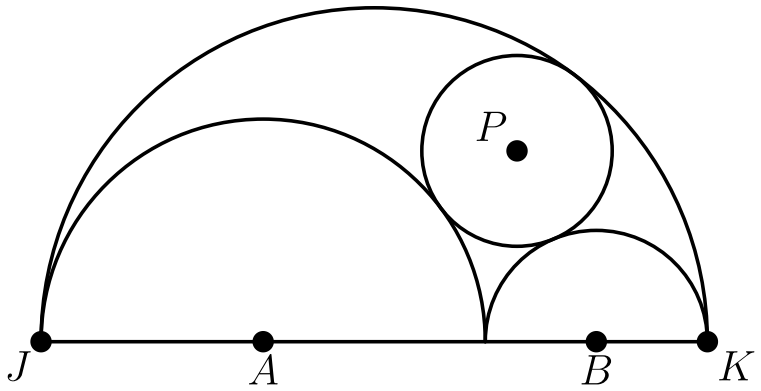
Thus no root occurs before $x = \pi$.

At $x = \pi$, $f(\pi) = 0 + 2(-1) + 0 = -2 < 0$. At $x = \dfrac{5\pi}{4}$, $\tan x = 1$ so $f = -\dfrac{\sqrt{2}}{2} + 3 > 0$. By the intermediate value theorem the smallest positive root lies in $\left(\pi, \dfrac{5\pi}{4}\right)$.

Since $\pi > 3$ and $\dfrac{5\pi}{4} < 4$, this interval sits inside $(3, 4)$.

Thus, the correct answer is **D**.

16. In the figure below, semicircles with centers at A and B and with radii 2 and 1, respectively, are drawn in the interior of, and sharing bases with, a semicircle with diameter $\overline{JK}$. The two smaller semicircles are externally tangent to each other and internally tangent to the largest semicircle. A circle centered at P is drawn externally tangent to the two smaller semicircles and internally tangent to the largest semicircle. What is the radius of the circle centered at P?



A $\frac{3}{4}$

B $\frac{6}{7}$

C $\frac{1}{2}\sqrt{3}$

D $\frac{5}{8}\sqrt{2}$

E $\frac{11}{12}$

Solution:

The large semicircle has radius 3 and center C, the midpoint of $\overline{JK}$. Placing J at the origin, $A = 2, B = 5, C = 3, K = 6$ along the base. Let r be the radius of the circle at P.

By tangency, $PA = 2 + r, PB = 1 + r$, and $PC = 3 - r$. Dropping a perpendicular from P to the base at horizontal position $3 + x$ with height h, the Pythagorean

theorem gives

$$\begin{aligned} h^2 &= (2+r)^2 - (1+x)^2 \\ &= (3-r)^2 - x^2 \\ &= (1+r)^2 - (2-x)^2. \end{aligned}$$

Equating the first expression with the middle one gives $5r - x = 3$, while equating the last expression with the middle one gives $2r + x = 3$. Adding yields $7r = 6$, so $r = \frac{6}{7}$.

Thus, the correct answer is **B**.

17. There are 24 different complex numbers z such that $z^{24} = 1$. For how many of these is z^6 a real number?

A 0

B 4

C 6

D 12

E 24

Solution:

The 24 solutions are the 24th roots of unity, $z = e^{\frac{\pi i k}{12}}$ for $k = 0, 1, \ldots, 23$.

Then $z^6 = e^{\frac{\pi i k}{2}} = \cos \frac{k\pi}{2} + i \sin \frac{k\pi}{2}$, which is real exactly when $\sin \frac{k\pi}{2} = 0$, i.e. when k is even. There are 12 even values of k in the range.

Thus, the correct answer is **D**.

18. Let $S(n)$ equal the sum of the digits of positive integer n. For example, $S(1507) = 13$. For a particular positive integer n, $S(n) = 1274$. Which of the following could be the value of $S(n+1)$?

A 1

B 3

C 12

D 1239

E 1265

Solution:

Adding 1 to n increases the digit sum by 1, except that each trailing 9 turns into a 0, losing 9. If n ends in exactly k nines, then $S(n+1) = S(n) + 1 - 9k = 1275 - 9k$.

So the possible values are $1275, 1266, 1257, \ldots$ Among the choices, only $1239 = 1275 - 9 \cdot 4$ fits (for example, n ending in four 9s preceded by enough 1s).

Thus, the correct answer is **D**.

19. A square with side length x is inscribed in a right triangle with sides of length $3, 4$, and 5 so that one vertex of the square coincides with the right-angle vertex of the triangle. A square with side length y is inscribed in another right triangle with sides of length $3, 4$, and 5 so that one side of the square lies on the hypotenuse of the triangle. What is $\frac{x}{y}$?

A $\frac{12}{13}$

B $\frac{35}{37}$

C 1

D $\frac{37}{35}$

E $\frac{13}{12}$

Solution:

For the first square, the two smaller triangles it cuts off are similar to the whole triangle, giving $\frac{x}{3-x} = \frac{4-x}{x}$, so $x = \frac{12}{7}$. (Equivalently, a square in the right angle has side $\frac{3 \cdot 4}{3+4}$.)

For the second square, take the hypotenuse of length 5 as base; the altitude to it is $h = \frac{3 \cdot 4}{5} = \frac{12}{5}$. A square with a side on a base b and height h has side $\frac{bh}{b+h}$, so

$$y = \frac{5 \cdot \frac{12}{5}}{5 + \frac{12}{5}} = \frac{12}{\frac{37}{5}} = \frac{60}{37}.$$

Therefore $\frac{x}{y} = \frac{12}{7} \cdot \frac{37}{60} = \frac{37}{35}$.

Thus, the correct answer is **D**.

20. How many ordered pairs (a, b) such that a is a positive real number and b is an integer between 2 and 200, inclusive, satisfy the equation $(\log_b a)^{2017} = \log_b(a^{2017})$?

A 198

B 199

C 398

D 399

E 597

Solution:

Let $u = \log_b a$. Since $\log_b(a^{2017}) = 2017 \log_b a$, the equation is $u^{2017} = 2017u$, so $u = 0$ or $u^{2016} = 2017$.

If $u = 0$, then $a = 1$, valid for every one of the 199 bases. If $u^{2016} = 2017$, then $u = \pm 2017^{\frac{1}{2016}}$, giving 2 values of a for each base, i.e. $2 \cdot 199 = 398$ pairs.

In total there are $199 + 398 = 597$ ordered pairs.

Thus, the correct answer is **E**.

21. A set S is constructed as follows. To begin, $S = \{0, 10\}$. Repeatedly, as long as possible, if x is an integer root of some polynomial $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ for some $n \geq 1$, all of whose coefficients a_i are elements of S, then x is put into S. When no more elements can be added to S, how many elements does S have?

A 4

B 5

C 7

D 9

E 11

Solution:

Using $10x + 10$, the root -1 enters S. Then 1 enters as a root of $-x^{10} - x^9 - \cdots - x + 10$, and -10 enters from $x + 10$.

Now $x^3 + x - 10$ has root 2, and $x + 2$ gives -2; then $2x - 10$ and $2x + 10$ give ± 5. At this point $S = \{0, \pm 1, \pm 2, \pm 5, \pm 10\}$.

No further integer can appear. Inductively, every nonzero member of S divides 10. If a polynomial used in the rule has constant term 0, factor out the largest possible power of x; any nonzero root is then a root of a polynomial whose constant term is the first nonzero original coefficient. The Rational Root Theorem shows that the root divides this coefficient, which by the inductive hypothesis divides 10. So S has 9 elements.

Thus, the correct answer is **D**.

22. A square is drawn in the Cartesian coordinate plane with vertices at $(2, 2)$, $(-2, 2)$, $(-2, -2)$, and $(2, -2)$. A particle starts at $(0, 0)$. Every second it moves with equal probability to one of the eight lattice points closest to its current position, independently of its previous moves. In other words, the probability is $\frac{1}{8}$ that the particle will move from (x, y) to each of $(x, y + 1)$, $(x + 1, y + 1)$, $(x + 1, y)$, $(x + 1, y - 1)$, $(x, y - 1)$, $(x - 1, y - 1)$, $(x - 1, y)$, or $(x - 1, y + 1)$. The particle will eventually hit the square for the first time, either at one of the 4 corners of the square or at one of the 12 lattice points in the interior of one of the sides of the square. The probability that it will hit at a corner rather than at an interior point of a side is $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n$?

A 4

B 5

C 7

D 15

E 39

Solution:

By symmetry, group the relevant interior points into three types: $C = \{(0, 0)\}$, the "axis" points $A = \{(\pm 1, 0), (0, \pm 1)\}$, and the "diagonal" points $I = \{(\pm 1, \pm 1)\}$. Let a, c, i be the probabilities of eventually hitting a corner starting from a point of type A, C, I.

Reading off the transition probabilities (a point in A goes to A with prob $\frac{2}{8}$, to C with $\frac{1}{8}$, to I with $\frac{2}{8}$, and to a side interior with $\frac{3}{8}$, etc.) gives

$$\begin{aligned} a &= \tfrac{2}{8}a + \tfrac{1}{8}c + \tfrac{2}{8}i, \\ c &= \tfrac{4}{8}a + \tfrac{4}{8}i, \\ i &= \tfrac{2}{8}a + \tfrac{1}{8}c + \tfrac{1}{8}. \end{aligned}$$

Solving yields $a = \frac{1}{14}, c = \frac{4}{35}, i = \frac{11}{70}$. The required probability is $c = \frac{4}{35}$, so $m + n = 4 + 35 = 39$.

Thus, the correct answer is **E**.

23. For certain real numbers a, b, and c, the polynomial

$$g(x) = x^3 + ax^2 + x + 10$$

has three distinct roots, and each root of $g(x)$ is also a root of the polynomial

$$f(x) = x^4 + x^3 + bx^2 + 100x + c.$$

What is $f(1)$?

A -9009

B -8008

C -7007

D -6006

E -5005

Solution:

Since g has three distinct roots all shared by the quartic f, we can write $f(x) = (x - q)g(x)$ for some remaining root q. Expanding,

$$f(x) = x^4 + (a - q)x^3 + (1 - qa)x^2 + (10 - q)x - 10q.$$

Matching the x coefficient, $10 - q = 100$, so $q = -90$. Matching the x^3 coefficient, $a - q = 1$, so $a = -89$.

Then $g(1) = 1 + a + 1 + 10 = 12 - 89 = -77$ and $1 - q = 91$, so

$$f(1) = (1 - q)g(1) = 91 \cdot (-77) = -7007.$$

Thus, the correct answer is **C**.

24. Quadrilateral $ABCD$ is inscribed in circle O and has sides $AB = 3, BC = 2, CD = 6$, and $DA = 8$. Let X and Y be points on BD such that $\frac{DX}{BD} = \frac{1}{4}$ and $\frac{BY}{BD} = \frac{11}{36}$. Let E be the intersection of line AX and the line through Y parallel to AD. Let F be the intersection of line CX and the line through E parallel to AC. Let G be the point on circle O other than C that lies on line CX. What is $XF \cdot XG$?

A 17

B $\frac{59 - 5\sqrt{2}}{3}$

C $\frac{91 - 12\sqrt{3}}{4}$

D $\frac{67 - 10\sqrt{2}}{3}$

E 18

Solution:

Because $YE \parallel AD$ and $EF \parallel AC$, we get $\triangle XEY \sim \triangle XAD$ and $\triangle XEF \sim \triangle XAC$, giving $\frac{XY}{XE} = \frac{XD}{XA}$ and $\frac{XF}{XE} = \frac{XC}{XA}$. Hence $\frac{XC}{XD} = \frac{XF}{XY}$, so $XF \cdot XD = XC \cdot XY$.

Power of a Point at X gives $XC \cdot XG = XD \cdot XB$, and combining yields $XF \cdot XG = XB \cdot XY$. With $d = BD, DX = \frac{1}{4}d$ and $BY = \frac{11}{36}d$, so

$$\begin{aligned} XF \cdot XG &= \left(d - \tfrac{1}{4}d\right) \\ &\quad \cdot \left(d - \tfrac{1}{4}d - \tfrac{11}{36}d\right) \\ &= \frac{3}{4}d \cdot \frac{4}{9}d = \frac{d^2}{3}. \end{aligned}$$

Since $ABCD$ is cyclic, $\angle BAD$ and $\angle BCD$ are supplementary. The Law of Cosines on $\triangle ABD$ and $\triangle CBD$ gives $\frac{73 - d^2}{48} = \frac{d^2 - 40}{24}$, so $d^2 = 51$. Therefore $XF \cdot XG = \frac{51}{3} = 17$.

Thus, the correct answer is **A**.

25. The vertices V of a centrally symmetric hexagon in the complex plane are given by

$$V = \left\{ \begin{array}{l} \sqrt{2}i,\ -\sqrt{2}i, \\ \frac{1}{\sqrt{8}}(1+i), \\ \frac{1}{\sqrt{8}}(-1+i), \\ \frac{1}{\sqrt{8}}(1-i), \\ \frac{1}{\sqrt{8}}(-1-i) \end{array} \right\}.$$

For each $j, 1 \leq j \leq 12$, an element z_j is chosen from V at random, independently of the other choices. Let $P = \prod_{j=1}^{12} z_j$ be the product of the 12 numbers selected. What is the probability that $P = -1$?

A $\quad \dfrac{5 \cdot 11}{3^{10}}$

B $\quad \dfrac{5^2 \cdot 11}{2 \cdot 3^{10}}$

C $\quad \dfrac{5 \cdot 11}{3^9}$

D $\quad \dfrac{5 \cdot 7 \cdot 11}{2 \cdot 3^{10}}$

E $\quad \dfrac{2^2 \cdot 5 \cdot 11}{3^{10}}$

Solution:

Let $A = \{\sqrt{2}i, -\sqrt{2}i\}$ (each of magnitude $\sqrt{2}$) and B be the other four elements (each of magnitude $\dfrac{1}{2}$). Since $|P| = (\sqrt{2})^{\#A} \left(\frac{1}{2}\right)^{\#B} = 1$ forces $\#A = 8$ and $\#B = 4$, exactly 8 factors must come from A and 4 from B.

A product of 8 elements of A equals ± 16 (real), and a product of 4 elements of B equals one of $\pm\frac{1}{16}, \pm\frac{i}{16}$. Their product is one of $\pm 1, \pm i$, each equally likely, so exactly $\frac{1}{4}$ of these configurations give $P = -1$.

The chance of landing in the 8-from-A, 4-from-B pattern is $\binom{12}{4}\left(\frac{1}{3}\right)^8\left(\frac{2}{3}\right)^4 = \dfrac{880}{3^{10}}$. Multiplying by $\frac{1}{4}$ gives

$$\begin{aligned} P &= \frac{1}{4} \cdot \frac{880}{3^{10}} \\ &= \frac{220}{3^{10}} = \frac{2^2 \cdot 5 \cdot 11}{3^{10}}. \end{aligned}$$

Thus, the correct answer is **E**.

Problems: https://live.poshenloh.com/past-contests/amc12/2017A

2017 AMC 12B Solutions

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1. Kymbrea's comic book collection currently has 30 comic books in it, and she is adding to her collection at the rate of 2 comic books per month. LaShawn's collection currently has 10 comic books in it, and he is adding to his collection at the rate of 6 comic books per month. After how many months will LaShawn's collection have twice as many comic books as Kymbrea's?

A 1

B 4

C 5

D 20

E 25

Solution:

After m months, Kymbrea has $30 + 2m$ comic books and LaShawn has $10 + 6m$. Setting $10 + 6m = 2(30 + 2m)$ gives $10 + 6m = 60 + 4m$, so $2m = 50$ and $m = 25$.

Thus, the correct answer is **E**.

2. Real numbers x, y, and z satisfy the inequalities

$$0 < x < 1, \quad -1 < y < 0,$$

and

$$1 < z < 2.$$

Which of the following numbers is necessarily positive?

A $y + x^2$

B $y + xz$

C $y + y^2$

D $y + 2y^2$

E $y + z$

Solution:

Adding $y > -1$ and $z > 1$ gives $y + z > 0$, so $y + z$ is always positive. Each of the other four choices can be made negative: with $x = \frac{1}{8}, y = -\frac{1}{4}, z = \frac{3}{2}$, every one of $y + x^2, y + xz, y + y^2$, and $y + 2y^2$ is negative.

Thus, the correct answer is **E**.

3. Suppose that x and y are nonzero real numbers such that

$$\frac{3x+y}{x-3y} = -2.$$

What is the value of

$$\frac{x+3y}{3x-y}?$$

A -3

B -1

C 1

D 2

E 3

Solution:

The equation gives $3x + y = -2(x - 3y) = -2x + 6y$, so $5x = 5y$, meaning $x = y$. Then

$$\frac{x+3y}{3x-y} = \frac{y+3y}{3y-y} = \frac{4y}{2y} = 2.$$

Thus, the correct answer is **D**.

4. Samia set off on her bicycle to visit her friend, traveling at an average speed of 17 kilometers per hour. When she had gone half the distance to her friend's house, a tire went flat, and she walked the rest of the way at 5 kilometers per hour. In all it took her 44 minutes to reach her friend's house. In kilometers rounded to the nearest tenth, how far did Samia walk?

A 2.0

B 2.2

C 2.8

D 3.4

E 4.4

Solution:

Let $2d$ be the total distance, so she biked d at 17 km/h and walked d at 5 km/h. The total time in hours is

$$\frac{d}{17} + \frac{d}{5} = \frac{44}{60}.$$

Combining the left side gives $\frac{22d}{85} = \frac{11}{15}$, so $d = \frac{11}{15} \cdot \frac{85}{22} = \frac{17}{6} = 2.833\ldots$ She walked about 2.8 kilometers.

Thus, the correct answer is **C**.

5. The data set $[6, 19, 33, 33, 39, 41, 41, 43, 51, 57]$ has median $Q_2 = 40$, first quartile $Q_1 = 33$, and third quartile $Q_3 = 43$. An outlier in a data set is a value that is more than 1.5 times the interquartile range below the first quartile (Q_1) or more than 1.5 times the interquartile range above the third quartile (Q_3), where the interquartile range is defined as $Q_3 - Q_1$. How many outliers does this data set have?

A 0

B 1

C 2

D 3

E 4

Solution:

The interquartile range is $43 - 33 = 10$, so 1.5 times it is 15. Outliers are values less than $33 - 15 = 18$ or greater than $43 + 15 = 58$. Only 6 falls below 18, and nothing exceeds 58, so there is exactly 1 outlier.

Thus, the correct answer is **B**.

6. The circle having $(0, 0)$ and $(8, 6)$ as the endpoints of a diameter intersects the x-axis at a second point. What is the x-coordinate of this point?

A $4\sqrt{2}$

B 6

C $5\sqrt{2}$

D 8

E $6\sqrt{2}$

Solution:

The center is the midpoint of the diameter, $(4, 3)$, and the radius is $\sqrt{4^2 + 3^2} = 5$. The circle is $(x - 4)^2 + (y - 3)^2 = 25$. Setting $y = 0$ gives $(x - 4)^2 = 16$, so $x = 0$ or $x = 8$. The second intersection with the x-axis is at $x = 8$.

Thus, the correct answer is **D**.

7. The functions $\sin(x)$ and $\cos(x)$ are periodic with least period 2π. What is the least period of the function $\cos(\sin(x))$?

A $\frac{\pi}{2}$

B π

C 2π

D 4π

E It's not periodic.

Solution:

Since $\cos(\sin(x+\pi)) = \cos(-\sin(x)) = \cos(\sin(x))$, the function has period π. It cannot be smaller: $\cos(\sin(x)) = 1$ exactly when $\sin(x) = 0$, which happens only at integer multiples of π, so the maxima are spaced π apart. The least period is π.

Thus, the correct answer is **B**.

8. The ratio of the short side of a certain rectangle to the long side is equal to the ratio of the long side to the diagonal. What is the square of the ratio of the short side to the long side of this rectangle?

A $\frac{\sqrt{3}-1}{2}$

B $\frac{1}{2}$

C $\frac{\sqrt{5}-1}{2}$

D $\frac{\sqrt{2}}{2}$

E $\frac{\sqrt{6}-1}{2}$

Solution:

Let x and y be the short and long sides, so the diagonal is $\sqrt{x^2+y^2}$ and $\frac{x^2}{y^2} = \frac{y^2}{x^2+y^2}$. Writing $r = \frac{x^2}{y^2}$, the right side is $\frac{y^2}{x^2+y^2} = \frac{1}{r+1}$, so $r = \frac{1}{r+1}$, giving $r^2 + r - 1 = 0$. The positive root is $r = \frac{\sqrt{5}-1}{2}$.

Thus, the correct answer is **C**.

9. A circle has center $(-10, -4)$ and radius 13. Another circle has center $(3, 9)$ and radius $\sqrt{65}$. The line passing through the two points of intersection of the two circles has equation $x + y = c$. What is c?

A 3

B $3\sqrt{3}$

C $4\sqrt{2}$

D 6

E $\frac{13}{2}$

Solution:

The circles are $(x + 10)^2 + (y + 4)^2 = 169$ and $(x - 3)^2 + (y - 9)^2 = 65$. Expanding and subtracting the second from the first cancels the x^2 and y^2 terms and simplifies to $x + y = 3$. Any intersection point satisfies this, so it is the line through both, and $c = 3$.

Thus, the correct answer is **A**.

10. At Typico High School, 60% of the students like dancing, and the rest dislike it. Of those who like dancing, 80% say that they like it, and the rest say that they dislike it. Of those who dislike dancing, 90% say that they dislike it, and the rest say that they like it. What fraction of students who say they dislike dancing actually like it?

A 10%

B 12%

C 20%

D 25%

E $33\frac{1}{3}\%$

Solution:

Students who like dancing but say they dislike it make up $60\% \cdot 20\% = 12\%$ of all students. Students who dislike dancing and say so make up $40\% \cdot 90\% = 36\%$. Among everyone who says they dislike dancing, the fraction who actually like it is

$$\frac{12}{12+36} = \frac{12}{48} = \frac{1}{4} = 25\%.$$

Thus, the correct answer is **D**.

11. Call a positive integer *monotonous* if it is a one-digit number or its digits, when read from left to right, form either a strictly increasing or a strictly decreasing sequence. For example, $3, 23578,$ and 987620 are monotonous, but $88, 7434,$ and 23557 are not. How many monotonous positive integers are there?

A 1024

B 1524

C 1533

D 1536

E 2048

Solution:

Strictly increasing monotonous numbers correspond to nonempty subsets of $\{1, \ldots, 9\}$, giving $2^9 - 1 = 511$. Strictly decreasing ones correspond to subsets of $\{0, 1, \ldots, 9\}$ other than $\varnothing$ and $\{0\}$ (a leading 0 is not allowed), giving $2^{10} - 2 = 1022$. The nine single-digit numbers are counted in both, so the total is $511 + 1022 - 9 = 1524$.

Thus, the correct answer is **B**.

12. What is the sum of the roots of $z^{12} = 64$ that have a positive real part?

A 2

B 4

C $2 + 2\sqrt{3}$

D $2\sqrt{2} + \sqrt{6}$

E $(1 + \sqrt{3}) + (1 + \sqrt{3})i$

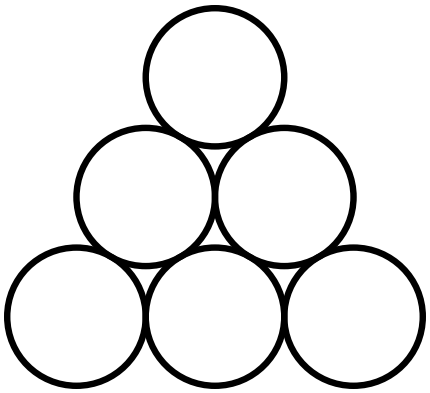
Solution:

The roots of $z^{12} = 64$ lie on the circle of radius $64^{\frac{1}{12}} = \sqrt{2}$, at angles that are multiples of 30°. Those with positive real part are at angles $0, \pm 30^\circ, \pm 60^\circ$. Their imaginary parts cancel, so the sum is

$$\begin{aligned} &\sqrt{2} + 2\sqrt{2}\cos 30^\circ \\ &\quad + 2\sqrt{2}\cos 60^\circ \\ &\quad = \sqrt{2}\left(1 + \sqrt{3} + 1\right) \\ &\quad = 2\sqrt{2} + \sqrt{6}. \end{aligned}$$

Thus, the correct answer is **D**.

13. In the figure below, 3 of the 6 disks are to be painted blue, 2 are to be painted red, and 1 is to be painted green. Two paintings that can be obtained from one another by a rotation or a reflection of the entire figure are considered the same. How many different paintings are possible?



A 6

B 8

C 9

D 12

E 15

Solution:

Before accounting for symmetry, there are $\frac{6!}{3!2!} = 60$ paintings. The two nonidentity rotations partition the disks into two 3-cycles, so neither can fix a painting having color counts $3, 2, 1$.

Each of the 3 reflections fixes 2 disks and swaps the other 4 in 2 pairs. For a painting to be fixed, the lone green disk and one of the 3 blue disks must occupy the two fixed positions, in 2 orders. Of the two swapped pairs, either one can be the red pair, giving $2 \cdot 2 = 4$ fixed paintings per reflection. Burnside's Lemma therefore gives $\frac{60 + 3(4)}{6} = 12$ distinct paintings.

Thus, the correct answer is **D**.

14. An ice-cream novelty item consists of a cup in the shape of a 4-inch-tall frustum of a right circular cone, with a 2-inch-diameter base at the bottom and a 4-inch-diameter base at the top, packed solid with ice cream, together with a solid cone of ice cream of height 4 inches, whose base, at the bottom, is the top base of the frustum. What is the total volume of the ice cream, in cubic inches?

A 8π

B $\frac{28\pi}{3}$

C 12π

D 14π

E $\frac{44\pi}{3}$

Solution:

Extending the frustum's sides to a point, similar triangles show the frustum equals a cone of radius 2 and height 8 minus a cone of radius 1 and height 4 :

$$\begin{aligned} &\tfrac{1}{3}\pi(2^2)(8) \\ &\quad -\tfrac{1}{3}\pi(1^2)(4) \\ &\quad =\tfrac{32}{3}\pi-\tfrac{4}{3}\pi \\ &\quad =\tfrac{28}{3}\pi. \end{aligned}$$

The top cone of radius 2 and height 4 adds $\frac{1}{3}\pi(2^2)(4)=\frac{16}{3}\pi$. The total is $\frac{28}{3}\pi+\frac{16}{3}\pi=\frac{44}{3}\pi$.

Thus, the correct answer is **E**.

15. Let ABC be an equilateral triangle. Extend side $\overline{AB}$ beyond B to a point B' so that $BB' = 3 \cdot AB$. Similarly, extend side $\overline{BC}$ beyond C to a point C' so that $CC' = 3 \cdot BC$, and extend side $\overline{CA}$ beyond A to a point A' so that $AA' = 3 \cdot CA$. What is the ratio of the area of $\triangle A'B'C'$ to the area of $\triangle ABC$?

A $9 : 1$

B $16 : 1$

C $25 : 1$

D $36 : 1$

E $37 : 1$

Solution:

Let $X = [\triangle ABC]$, and draw segments CB', AC', and BA'. Triangle $BB'C$ has base $BB' = 3 \cdot AB$ and the same altitude as $\triangle ABC$ from C to line AB, so its area is $3X$; likewise $\triangle CC'A$ and $\triangle AA'B$ each have area $3X$. Next, $\triangle AA'C'$ has 3 times the base and the same height as $\triangle ACC'$, so its area is $9X$; similarly $\triangle CC'B'$ and $\triangle BB'A'$ each have area $9X$. Thus

$$\begin{aligned}[\triangle A'B'C'] &= X \\ &\quad + 3(3X) + 3(9X) \\ &= 37X,\end{aligned}$$

so the ratio is $37 : 1$.

Thus, the correct answer is **E**.

16. The number $21! = 51{,}090{,}942{,}171{,}709{,}440{,}000$ has over $60{,}000$ positive integer divisors. One of them is chosen at random. What is the probability that it is odd?

A $\frac{1}{21}$

B $\frac{1}{19}$

C $\frac{1}{18}$

D $\frac{1}{2}$

E $\frac{11}{21}$

Solution:

The exponent of 2 in $21!$ is $\lfloor \frac{21}{2} \rfloor + \lfloor \frac{21}{4} \rfloor + \lfloor \frac{21}{8} \rfloor + \lfloor \frac{21}{16} \rfloor = 10 + 5 + 2 + 1 = 18$. Every divisor has the form $2^i b$ with $0 \leq i \leq 18$ and b odd; it is odd exactly when $i = 0$. So the fraction of odd divisors is $\frac{1}{18+1} = \frac{1}{19}$.

Thus, the correct answer is **B**.

17. A coin is biased in such a way that on each toss the probability of heads is $\frac{2}{3}$ and the probability of tails is $\frac{1}{3}$. The outcomes of the tosses are independent. A player has the choice of playing Game A or Game B. In Game A she tosses the coin three times and wins if all three outcomes are the same. In Game B she tosses the coin four times and wins if both the outcomes of the first and second tosses are the same and the outcomes of the third and fourth tosses are the same. How do the chances of winning Game A compare to the chances of winning Game B?

A The probability of winning Game A is $\frac{4}{81}$ less than the probability of winning Game B.

B The probability of winning Game A is $\frac{2}{81}$ less than the probability of winning Game B.

C The probabilities are the same.

D The probability of winning Game A is $\frac{2}{81}$ greater than the probability of winning Game B.

E The probability of winning Game A is $\frac{4}{81}$ greater than the probability of winning Game B.

Solution:

Let $p = \frac{2}{3}$. Game A is won when all three tosses match: $p^3 + (1-p)^3$. Game B needs the first pair to match and the second pair to match, each with probability $p^2 + (1-p)^2$, so the win probability is $\left(p^2 + (1-p)^2\right)^2$. With $p = \frac{2}{3}$, Game A gives $\left(\frac{2}{3}\right)^3 + \left(\frac{1}{3}\right)^3 = \frac{9}{27} = \frac{1}{3}$, and Game B gives $\left(\frac{4}{9} + \frac{1}{9}\right)^2 = \left(\frac{5}{9}\right)^2 = \frac{25}{81}$. The difference is $\frac{27}{81} - \frac{25}{81} = \frac{2}{81}$, so Game A is $\frac{2}{81}$ more likely.

Thus, the correct answer is **D**.

18. The diameter $\overline{AB}$ of a circle of radius 2 is extended to a point D outside the circle so that $BD = 3$. Point E is chosen so that $ED = 5$ and line ED is perpendicular to line AD. Segment $\overline{AE}$ intersects the circle at a point C between A and E. What is the area of $\triangle ABC$?

A $\frac{120}{37}$

B $\frac{140}{39}$

C $\frac{145}{39}$

D $\frac{140}{37}$

E $\frac{120}{31}$

Solution:

Since $\angle ACB$ is inscribed in a semicircle, it is a right angle, so $\triangle ABC \sim \triangle AED$ (both right-angled and sharing angle A). Their areas are in ratio $AB^2 : AE^2$. Here $AB = 4$, so $AB^2 = 16$, and $AD = AB + BD = 7$, so $AE^2 = AD^2 + ED^2 = 49 + 25 = 74$. The area of $\triangle AED$ is $\frac{1}{2} \cdot 7 \cdot 5 = \frac{35}{2}$. Thus

$$[\triangle ABC] = \frac{16}{74} \cdot \frac{35}{2} = \frac{140}{37}.$$

Thus, the correct answer is **D**.

19. Let $N = 123456789101112\ldots4344$ be the 79-digit number that is formed by writing the integers from 1 to 44 in order, one after the other. What is the remainder when N is divided by 45?

A 1

B 4

C 9

D 18

E 44

Solution:

The last digit of N is 4, so $N \equiv 4 \pmod 5$. For mod 9, sum the digits: the numbers 1–9 contribute their digits, the tens digits of 10–44 and the units digits together sum to 270, which is a multiple of 9, so $N \equiv 0 \pmod 9$. The number $N - 9$ is then a multiple of 9, and its last digit is 5, so it is a multiple of 5; hence $N - 9$ is a multiple of 45. Therefore $N \equiv 9 \pmod{45}$.

Thus, the correct answer is **C**.

20. Real numbers x and y are chosen independently and uniformly at random from the interval $(0, 1)$. What is the probability that $\lfloor \log_2 x \rfloor = \lfloor \log_2 y \rfloor$, where $\lfloor r \rfloor$ denotes the greatest integer less than or equal to the real number r?

A $\frac{1}{8}$

B $\frac{1}{6}$

C $\frac{1}{4}$

D $\frac{1}{3}$

E $\frac{1}{2}$

Solution:

For each positive integer n, $\lfloor \log_2 x \rfloor = -n$ exactly when $\frac{1}{2^n} \leq x < \frac{1}{2^{n-1}}$, an interval of length $\frac{1}{2^n}$. The event that both floors equal $-n$ is a square of area $\frac{1}{4^n}$. Summing over all n, the probability is

$$\sum_{n=1}^{\infty} \frac{1}{4^n} = \frac{\frac{1}{4}}{1 - \frac{1}{4}} = \frac{1}{3}.$$

Thus, the correct answer is **D**.

21. Last year Isabella took 7 math tests and received 7 different scores, each an integer between 91 and 100, inclusive. After each test she noticed that the average of her test scores was an integer. Her score on the seventh test was 95. What was her score on the sixth test?

A 92

B 94

C 96

D 98

E 100

Solution:

Let S be the sum of all seven scores. Then S is a multiple of 7 with $658 \leq S \leq 679$, so $S \in \{658, 665, 672, 679\}$. Since the average after six tests is an integer, $S - 95$ is a multiple of 6, which forces $S = 665$. Then the first six scores sum to 570, a multiple of 5; the average after five tests is an integer, so the first five scores also sum to a multiple of 5, making the sixth score a multiple of 5. Since all scores differ and the seventh is 95, the sixth must be 100.

Thus, the correct answer is **E**.

22. Abby, Bernardo, Carl, and Debra play a game in which each of them starts with four coins. The game consists of four rounds. In each round, four balls are placed in an urn—one green, one red, and two white. The players each draw a ball at random without replacement. Whoever gets the green ball gives one coin to whoever gets the red ball. What is the probability that, at the end of the fourth round, each of the players has four coins?

A $\frac{7}{576}$

B $\frac{5}{192}$

C $\frac{1}{36}$

D $\frac{5}{144}$

E $\frac{7}{48}$

Solution:

Each round has $4 \cdot 3 = 12$ equally likely (giver, receiver) pairs, so there are 12^4 outcome sequences. Everyone ends with four coins exactly when the four transfers cancel. The favorable patterns are: a 4-cycle of gifts ($24 \cdot 6 = 144$ ways), two disjoint mutual exchanges ($24 \cdot 3 = 72$), one pair exchanging twice ($6 \cdot 6 = 36$), and one player both giving to and receiving from each of two others ($4 \cdot 3 \cdot 24 = 288$). These total $144 + 72 + 36 + 288 = 540$. The probability is $\frac{540}{12^4} = \frac{540}{20736} = \frac{5}{192}$.

Thus, the correct answer is **B**.

23. The graph of $y = f(x)$, where $f(x)$ is a polynomial of degree 3, contains points $A(2, 4), B(3, 9)$, and $C(4, 16)$. Lines AB, AC, and BC intersect the graph again at points D, E, and F, respectively, and the sum of the x-coordinates of D, E, and F is 24. What is $f(0)$?

A -2

B 0

C 2

D $\frac{24}{5}$

E 8

Solution:

The points A, B, C lie on $y = x^2$, so $g(x) = f(x) - x^2$ has roots $2, 3, 4 : g(x) = a(x-2)(x-3)(x-4)$ for some $a \neq 0$. The coefficients of x^3 and x^2 in f are a and $1 - 9a$, so by Vieta the three roots of $f(x) - L(x)$ (for any linear L) sum to $9 - \frac{1}{a}$. The lines AB, AC, BC meet the cubic in triples $\{2, 3, x_D\}, \{2, 4, x_E\}, \{3, 4, x_F\}$, so

$$\begin{aligned} x_D + x_E + x_F \\ &= 3\left(9 - \tfrac{1}{a}\right) \\ &\quad - 2(2 + 3 + 4) \\ &= 9 - \tfrac{3}{a} = 24, \end{aligned}$$

giving $a = -\frac{1}{5}$. Then $f(x) = x^2 - \frac{1}{5}(x-2)(x-3)(x-4)$, so $f(0) = 0 - \frac{1}{5}(-2)(-3)(-4) = \frac{24}{5}$.

Thus, the correct answer is **D**.

24. Quadrilateral $ABCD$ has right angles at B and C, $\triangle ABC \sim \triangle BCD$, and $AB > BC$. There is a point E in the interior of $ABCD$ such that $\triangle ABC \sim \triangle CEB$ and the area of $\triangle AED$ is 17 times the area of $\triangle CEB$. What is $\dfrac{AB}{BC}$?

A $1+\sqrt{2}$

B $2+\sqrt{2}$

C $\sqrt{17}$

D $2+\sqrt{5}$

E $1+2\sqrt{3}$

Solution:

Set $BC = 1$ and $AB = r > 1$. The similarity $\triangle ABC \sim \triangle BCD$ with the right angles places the figure at $C = (0,0), B = (0,1), A = (r,1), D = \left(\frac{1}{r}, 0\right)$. Let $E = (x,y)$ with $x, y > 0$. From $\triangle ABC \sim \triangle CEB$ we get $\dfrac{x}{y} = \tan(\angle ECB) = \tan(\angle BAC) = \dfrac{1}{r}$ and $x^2 + y^2 = \dfrac{r^2}{1+r^2}$, so $x = \dfrac{r}{1+r^2}, y = \dfrac{r^2}{1+r^2}$. The two relevant areas are

$$[\triangle CEB] = \frac{r}{2(1+r^2)},$$
$$[\triangle AED] = \frac{r^4 - r^2 + 1}{2r(1+r^2)}.$$

Setting the second equal to 17 times the first gives $r^4 - 18r^2 + 1 = 0$. Then $r^2 = 9 + 4\sqrt{5} = (2+\sqrt{5})^2$, so $r = 2+\sqrt{5}$.

Thus, the correct answer is **D**.

25. A set of n people participate in an online video basketball tournament. Each person may be a member of any number of 5-player teams, but no two teams may have exactly the same 5 members. The site statistics show a curious fact: The average, over all subsets of size 9 of the set of n participants, of the number of complete teams whose members are among those 9 people is equal to the reciprocal of the average, over all subsets of size 8 of the set of n participants, of the number of complete teams whose members are among those 8 people. How many values $n, 9 \leq n \leq 2017$, can be the number of participants?

A 477

B 482

C 487

D 557

E 562

Solution:

Let T be the number of teams. Summing over size-9 subsets counts each team $\binom{n-5}{4}$ times and over size-8 subsets $\binom{n-5}{3}$ times. The averages are $\frac{\binom{n-5}{4}T}{\binom{n}{9}}$ and $\frac{\binom{n-5}{3}T}{\binom{n}{8}}$; setting the first equal to the reciprocal of the second and simplifying gives

$$T = \frac{n(n-1)(n-2)(n-3)(n-4)}{2^5 \cdot 3^2 \cdot 5 \cdot 7}.$$

We need this to be a positive integer with $n \geq 9$. Let $N = n(n-1)(n-2)(n-3)(n-4)$; as a product of five consecutive integers, N is always divisible by 5. The condition $7 \mid N$ holds for 5 residues modulo 7; $9 \mid N$ holds for 7 residues modulo 9; and $32 \mid N$ holds for 8 residues modulo 16. The Chinese Remainder Theorem therefore gives $5 \cdot 7 \cdot 8 = 280$ solutions modulo $\operatorname{lcm}(7, 9, 16) = 1008$. So there are 560 values in $1 \leq n \leq 2016$; removing $n = 1, 2, 3, 4$ (which are below 9) and adding $n = 2017$ (since $2017 \equiv 1 \pmod{1008}$) gives $560 - 4 + 1 = 557$ valid values.

Thus, the correct answer is **D**.

Problems: https://live.poshenloh.com/past-contests/amc12/2017B