

AMC 12 SOLUTION ARCHIVE

2016

12A and 12B full explanations

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2016 AMC 12A Solutions

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1. What is the value of

$$\frac{11! - 10!}{9!}?$$

A 99

B 100

C 110

D 121

E 132

Solution:

Factoring the numerator gives

$$\begin{aligned} & \frac{11! - 10!}{9!} \\ &= \frac{10!(11 - 1)}{9!} \\ &= \frac{10 \cdot 9! \cdot 10}{9!} \\ &= 100. \end{aligned}$$

Thus, the correct answer is **B**.

2. For what value of x does $10^x \cdot 100^{2x} = 1000^5$?

A 1

B 2

C 3

D 4

E 5

Solution:

Since $100 = 10^2$ and $1000 = 10^3$, the equation becomes

$$10^x \cdot 10^{4x} = 10^{15},$$

so $10^{5x} = 10^{15}$. Then $5x = 15$, giving $x = 3$.

Thus, the correct answer is **C**.

3. The remainder function can be defined for all real numbers x and y with $y \neq 0$ by

$$\text{rem}(x, y) = x - y \left\lfloor \frac{x}{y} \right\rfloor,$$

where $\left\lfloor \frac{x}{y} \right\rfloor$ denotes the greatest integer less than or equal to $\frac{x}{y}$. What is the value of $\text{rem}\left(\frac{3}{8}, -\frac{2}{5}\right)$?

- A $-\frac{3}{8}$
- B $-\frac{1}{40}$**
- C 0
- D $\frac{3}{8}$
- E $\frac{31}{40}$

Solution:

First,

$$\begin{aligned}\frac{x}{y} &= \frac{\frac{3}{8}}{-\frac{2}{5}} \\ &= \frac{3}{8} \cdot \left(-\frac{5}{2}\right) \\ &= -\frac{15}{16},\end{aligned}$$

$$\text{and } \left\lfloor -\frac{15}{16} \right\rfloor = -1.$$

Therefore

$$\begin{aligned} & \text{rem} \left(\frac{3}{8}, -\frac{2}{5} \right) \\ &= \frac{3}{8} - \left(-\frac{2}{5} \right) (-1) \\ &= \frac{3}{8} - \frac{2}{5} \\ &= \frac{15 - 16}{40} \\ &= -\frac{1}{40}. \end{aligned}$$

Thus, the correct answer is **B**.

4. The mean, median, and mode of the 7 data values 60, 100, x , 40, 50, 200, 90 are all equal to x . What is the value of x ?

A 50

B 60

C 75

D 90

E 100

Solution:

The mean condition gives

$$\begin{aligned}\frac{60 + 100 + x + 40 + 50 + 200 + 90}{7} \\&= \frac{540 + x}{7} \\&= x,\end{aligned}$$

so $540 + x = 7x$ and $x = 90$.

In nondecreasing order the data are 40, 50, 60, 90, 90, 100, 200, so the median is 90 and the mode is 90, as required.

Thus, the correct answer is **D**.

5. Goldbach's conjecture states that every even integer greater than 2 can be written as the sum of two prime numbers (for example, $2016 = 13 + 2003$). So far, no one has been able to prove that the conjecture is true, and no one has found a counterexample to show that the conjecture is false. What would a counterexample consist of?

- ☐ A an odd integer greater than 2 that can be written as the sum of two prime numbers
- ☐ B an odd integer greater than 2 that cannot be written as the sum of two prime numbers
- ☐ C an even integer greater than 2 that can be written as the sum of two numbers that are not prime
- ☐ D an even integer greater than 2 that can be written as the sum of two prime numbers
- ☒ E an even integer greater than 2 that cannot be written as the sum of two prime numbers

Solution:

A counterexample must satisfy the hypothesis of being an even integer greater than 2 while failing the conclusion that it can be written as the sum of two prime numbers.

Thus, the correct answer is **E**.

6. A triangular array of 2016 coins has 1 coin in the first row, 2 coins in the second row, 3 coins in the third row, and so on up to N coins in the N th row. What is the sum of the digits of N ?

- A 6
- B 7
- C 8
- D 9**
- E 10

Solution:

The total number of coins is

$$\begin{aligned}1 + 2 + \cdots + N \\&= \frac{N(N + 1)}{2} \\&= 2016,\end{aligned}$$

so $N(N + 1) = 4032$. Since $63 \cdot 64 = 4032$, we have $N = 63$, and the sum of its digits is $6 + 3 = 9$.

Thus, the correct answer is **D**.

7. Which of these describes the graph of $x^2(x + y + 1) = y^2(x + y + 1)$?

A two parallel lines

B two intersecting lines

C three lines that all pass through a common point

D three lines that do not all pass through a common point

E a line and a parabola

Solution:

Moving all terms to one side gives

$$(x^2 - y^2)(x + y + 1) = 0,$$

which factors as

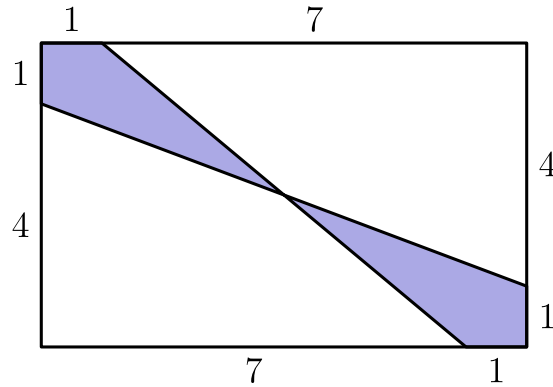
$$(x - y)(x + y)(x + y + 1) = 0.$$

The graph is therefore the union of the lines $x = y$, $x = -y$, and $x + y + 1 = 0$.

The first two lines intersect at the origin, but the third line $x + y = -1$ is parallel to $x + y = 0$ and does not pass through the origin. So the graph consists of three lines that do not all pass through a common point.

Thus, the correct answer is **D**.

8. What is the area of the shaded region of the given 8×5 rectangle?



A $4\frac{3}{4}$

B 5

C $5\frac{1}{4}$

D $6\frac{1}{2}$

E 8

Solution:

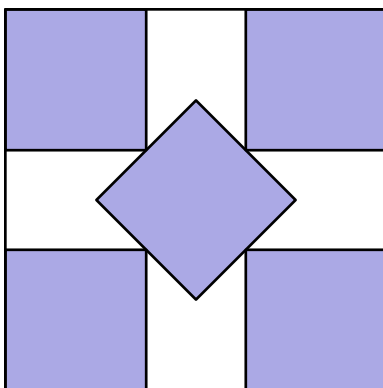
The diagonal of the rectangle from the upper-left corner to the lower-right corner divides the shaded region into four triangles, all meeting at the center of the rectangle.

Two of these triangles have a horizontal base of length 1 and altitude $\frac{1}{2} \cdot 5 = \frac{5}{2}$, and the other two have a vertical base of length 1 and altitude $\frac{1}{2} \cdot 8 = 4$. The total area is

$$\begin{aligned} & 2 \cdot \frac{1}{2} \cdot 1 \cdot \frac{5}{2} \\ & + 2 \cdot \frac{1}{2} \cdot 1 \cdot 4 \\ & = \frac{5}{2} + 4 \\ & = \frac{13}{2}. \end{aligned}$$

Thus, the correct answer is **D**.

9. The five small shaded squares inside this unit square are congruent and have disjoint interiors. The midpoint of each side of the middle square coincides with one of the vertices of the other four small squares as shown. The common side length is $\frac{a - \sqrt{2}}{b}$, where a and b are positive integers. What is $a + b$?



- ☐ A 7
- ☐ B 8
- ☐ C 9
- ☐ D 10
- ☒ E 11

Solution:

Let x be the common side length. The diagonal of the unit square has length $\sqrt{2}$ and consists of two small-square diagonals (each $x\sqrt{2}$) plus one small-square side length x , so

$$2x\sqrt{2} + x = \sqrt{2}.$$

Solving,

$$\begin{aligned}x &= \frac{\sqrt{2}}{2\sqrt{2} + 1} \\&= \frac{\sqrt{2}(2\sqrt{2} - 1)}{(2\sqrt{2} + 1)(2\sqrt{2} - 1)} \\&= \frac{4 - \sqrt{2}}{7}.\end{aligned}$$

Thus $a = 4$, $b = 7$, and $a + b = 11$.

Thus, the correct answer is **E**.

- 10.** Five friends sat in a movie theater in a row containing 5 seats, numbered 1 to 5 from left to right. (The directions “left” and “right” are from the point of view of the people as they sit in the seats.) During the movie Ada went to the lobby to get some popcorn. When she returned, she found that Bea had moved two seats to the right, Ceci had moved one seat to the left, and Dee and Edie had switched seats, leaving an end seat for Ada. In which seat had Ada been sitting before she got up?

- ☐ A 1
- ☒ B 2
- ☐ C 3
- ☐ D 4
- ☐ E 5

Solution:

The net displacement of all five friends is zero. Dee and Edie swapped seats, so their movements cancel. Bea moved $+2$ and Ceci moved -1 , a net of $+1$, so Ada must move -1 to balance.

Ada returns to an end seat; since she moved one seat to the left, that seat must be seat 1, so she had been sitting in seat 2.

Thus, the correct answer is **B**.

11. Each of the 100 students in a certain summer camp can either sing, dance, or act. Some students have more than one talent, but no student has all three talents. There are 42 students who cannot sing, 65 students who cannot dance, and 29 students who cannot act. How many students have two of these talents?

A 16

B 25

C 36

D 49

E 64

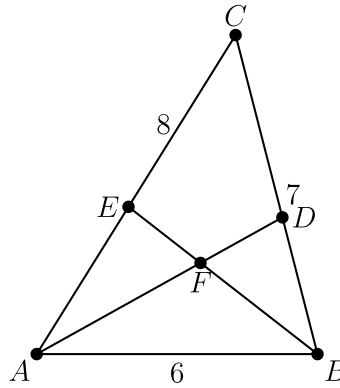
Solution:

The numbers who can sing, dance, and act are $100 - 42 = 58$, $100 - 65 = 35$, and $100 - 29 = 71$, respectively, for a total of $58 + 35 + 71 = 164$.

Since no student has all three talents, each student has one or two talents, so single-talent students are counted once and two-talent students are counted twice. The number counted twice is $164 - 100 = 64$.

Thus, the correct answer is **E**.

12. In $\triangle ABC$, $AB = 6$, $BC = 7$, and $CA = 8$. Point D lies on $\overline{BC}$, and $\overline{AD}$ bisects $\angle BAC$. Point E lies on $\overline{AC}$, and $\overline{BE}$ bisects $\angle ABC$. The bisectors intersect at F . What is the ratio $AF : FD$?



A 3 : 2

B 5 : 3

C 2 : 1

D 7 : 3

E 5 : 2

Solution:

Applying the Angle Bisector Theorem to $\triangle ABC$ gives $BD : DC = AB : AC = 6 : 8$, so $BD = \frac{6}{6+8} \cdot 7 = 3$.

Now BF lies along the bisector of $\angle ABD$ in $\triangle ABD$, so by the Angle Bisector Theorem again,

$$\begin{aligned} AF : FD &= AB : BD \\ &= 6 : 3 = 2 : 1. \end{aligned}$$

Thus, the correct answer is **C**.

13. Let N be a positive multiple of 5. One red ball and N green balls are arranged in a line in random order. Let $P(N)$ be the probability that at least $\frac{3}{5}$ of the green balls are on the same side of the red ball. Observe that $P(5) = 1$ and that $P(N)$ approaches $\frac{4}{5}$ as N grows large. What is the sum of the digits of the least value of N such that $P(N) < \frac{321}{400}$?

A 12

B 14

C 16

D 18

E 20

Solution:

Write $N = 5k$. Number the positions of the red ball $0, 1, \dots, 5k$ from one end; there are $5k + 1$ equally likely positions.

Fewer than $\frac{3}{5}$ of the green balls lie on each side exactly when the red ball is in one of the positions $2k + 1, 2k + 2, \dots, 3k - 1$, which is $k - 1$ positions. Hence

$$P(N) = 1 - \frac{k - 1}{5k + 1} = \frac{4k + 2}{5k + 1}.$$

Solving $\frac{4k + 2}{5k + 1} < \frac{321}{400}$ gives $400(4k + 2) < 321(5k + 1)$, so $1600k + 800 < 1605k + 321$ and $5k > 479$, meaning $k > 95.8$. Thus $k = 96$ and $N = 480$, whose digit sum is $4 + 8 + 0 = 12$.

Thus, the correct answer is **A**.

14. Each vertex of a cube is to be labeled with an integer from 1 through 8, with each integer being used once, in such a way that the sum of the four numbers on the vertices of a face is the same for each face. Arrangements that can be obtained from each other through rotations of the cube are considered to be the same. How many different arrangements are possible?

- A 1
- B 3
- C 6**
- D 12
- E 24

Solution:

Each vertex belongs to 3 faces, so $6S = 3(1 + 2 + \cdots + 8) = 108$, giving each face-sum $S = 18$.

The four-element subsets containing 1 with sum 18 are $\{1, 2, 7, 8\}$, $\{1, 3, 6, 8\}$, $\{1, 4, 5, 8\}$, and $\{1, 4, 6, 7\}$. Only one omits 8, so at least two of the three distinct faces through 1 contain 8. Two vertices of a cube lie on two common faces exactly when they are adjacent; hence 1 and 8 are adjacent.

Rotate the cube so that 1 is at the lower-left-front vertex and 8 at the lower-right-front vertex. The unique face through 1 that does not contain 8 must use 4, 6, 7, and these can be placed in $3! = 6$ orders. Each order forces 5, 3, 2 at the three opposite vertices, and the remaining face sums are then 18. Hence there are 6 arrangements.

Thus, the correct answer is **C**.

15. Circles with centers P , Q , and R , having radii 1, 2, and 3, respectively, lie on the same side of line l and are tangent to l at P' , Q' , and R' , respectively, with Q' between P' and R' . The circle with center Q is externally tangent to each of the other two circles. What is the area of $\triangle PQR$?

A 0

B $\sqrt{\frac{2}{3}}$

C 1

D $\sqrt{6} - \sqrt{2}$

E $\sqrt{\frac{3}{2}}$

Solution:

The centers lie at heights 1, 2, and 3 above line l . Since circle Q is externally tangent to circle P , we have $PQ = 3$, so the horizontal distance is $P'Q' = \sqrt{3^2 - 1^2} = \sqrt{8}$.

Since circle Q is tangent to circle R , we have $QR = 5$, so $Q'R' = \sqrt{5^2 - 1^2} = \sqrt{24}$.

Place $P = (0, 1)$, $Q = (\sqrt{8}, 2)$, and $R = (\sqrt{8} + \sqrt{24}, 3)$. By the shoelace formula, the area is

$$\begin{aligned} & \frac{1}{2} |\sqrt{8}(3 - 1) + (\sqrt{8} + \sqrt{24})(1 - 2)| \\ &= \frac{1}{2} (\sqrt{24} - \sqrt{8}) \\ &= \sqrt{6} - \sqrt{2}. \end{aligned}$$

Thus, the correct answer is **D**.

16. The graphs of $y = \log_3 x$, $y = \log_x 3$, $y = \log_{\frac{1}{3}} x$, and $y = \log_x \frac{1}{3}$ are plotted on the same set of axes. How many points in the plane with positive x -coordinates lie on two or more of the graphs?

A 2

B 3

C 4

D 5

E 6

Solution:

Let $u = \log_3 x$. Then $\log_x 3 = \frac{1}{u}$, $\log_{\frac{1}{3}} x = -u$, and $\log_x \frac{1}{3} = -\frac{1}{u}$. Two graphs meet where two of u , $\frac{1}{u}$, $-u$, $-\frac{1}{u}$ are equal for some valid $x > 0$.

Setting $u = \frac{1}{u}$ gives $u = \pm 1$, so $x = 3$ or $x = \frac{1}{3}$; setting $-u = -\frac{1}{u}$ gives the same values. Setting $u = -u$ gives $u = 0$, i.e. $x = 1$, where $\log_3 x$ and $\log_{\frac{1}{3}} x$ are both 0.

The remaining pairings have no real solution.

The distinct intersection points are $(1, 0)$, $(3, 1)$, $(\frac{1}{3}, -1)$, $(3, -1)$, and $(\frac{1}{3}, 1)$, so there are 5.

Thus, the correct answer is **D**.

17. Let $ABCD$ be a square. Let E, F, G , and H be the centers, respectively, of equilateral triangles with bases $\overline{AB}, \overline{BC}, \overline{CD}$, and $\overline{DA}$, each exterior to the square. What is the ratio of the area of square $EFGH$ to the area of square $ABCD$?

A 1

B $\frac{2 + \sqrt{3}}{3}$

C $\sqrt{2}$

D $\frac{\sqrt{2} + \sqrt{3}}{2}$

E $\sqrt{3}$

Solution:

Let square $ABCD$ have side length 6. Each equilateral triangle has height $3\sqrt{3}$, and its center lies $\frac{1}{3}$ of that height, namely $\sqrt{3}$, from the square's side.

Square $ABCD$ has diagonal $6\sqrt{2}$. Square $EFGH$ has diagonal equal to the side of $ABCD$ plus twice $\sqrt{3}$, namely $6 + 2\sqrt{3}$. The area ratio is the square of the ratio of diagonals:

$$\begin{aligned} & \left(\frac{6 + 2\sqrt{3}}{6\sqrt{2}} \right)^2 \\ &= \left(\frac{3 + \sqrt{3}}{3\sqrt{2}} \right)^2 \\ &= \frac{12 + 6\sqrt{3}}{18} \\ &= \frac{2 + \sqrt{3}}{3}. \end{aligned}$$

Thus, the correct answer is **B**.

18. For some positive integer n , the number $110n^3$ has 110 positive integer divisors, including 1 and the number $110n^3$. How many positive integer divisors does the number $81n^4$ have?

A 110

B 191

C 261

D 325

E 425

Solution:

The number $110n^3$ is divisible by the three distinct primes 2, 5, 11. If its prime exponents are $r_1, r_2, \dots$, then $(r_1 + 1)(r_2 + 1) \cdots = 110$, where $110 = 2 \cdot 5 \cdot 11$. Because there are already at least three factors greater than 1, there are exactly three, so no other prime divides n .

Each exponent in $110n^3$ is 1 (mod 3), so each divisor-count factor $r_i + 1$ is 2 (mod 3). The factors 2, 5, 11 all have that form, so the exponents are 1, 4, 10 in some order. After subtracting the exponent 1 from 110 and dividing by 3, the exponents in n are 0, 1, 3 in some order. Thus $n = pq^3$ for two distinct primes p, q chosen from 2, 5, 11.

Consequently n^4 has exponents 4 and 12, while $81 = 3^4$ introduces a third prime because $3 \nmid n$. Hence the number of divisors of $81n^4$ is

$$\begin{aligned} & (4 + 1)(4 + 1)(12 + 1) \\ &= 5 \cdot 5 \cdot 13 \\ &= 325. \end{aligned}$$

Thus, the correct answer is **D**.

19. Jerry starts at 0 on the real number line. He tosses a fair coin 8 times. When he gets heads, he moves 1 unit in the positive direction; when he gets tails, he moves 1 unit in the negative direction. The probability that he reaches 4 at some time during this process is $\frac{a}{b}$, where a and b are relatively prime positive integers. What is $a + b$? (For example, he succeeds if his sequence of tosses is HTHHHHHH.)

A 69

B 151

C 257

D 293

E 313

Solution:

Count the sequences of 8 tosses whose running total reaches 4. With at most 2 tails he certainly reaches 4, contributing

$$\begin{aligned}\binom{8}{0} + \binom{8}{1} + \binom{8}{2} \\&= 1 + 8 + 28 \\&= 37\end{aligned}$$

sequences.

With exactly 3 tails, he can first reach 4 on toss 4 or toss 6. Reaching it on toss 4 requires four initial heads, after which the remaining head has 4 possible positions. Otherwise, tosses 5 and 6 are heads, one of the first four tosses is a tail, and the last two are tails, again giving 4 possibilities. Thus this case contributes 8. With exactly 4 tails, only HHHHTTTT works, giving 1. He cannot reach 4 with fewer than 4 heads.

So there are $37 + 8 + 1 = 46$ favorable sequences out of $2^8 = 256$, a probability of $\frac{46}{256} = \frac{23}{128}$. Then $a + b = 23 + 128 = 151$.

Thus, the correct answer is **B**.

20. A binary operation $\diamond$ has the properties that $a \diamond (b \diamond c) = (a \diamond b) \cdot c$ and that $a \diamond a = 1$ for all nonzero real numbers a, b , and c . (Here the dot $\cdot$ represents the usual multiplication operation.) The solution to the equation $2016 \diamond (6 \diamond x) = 100$ can be written as $\frac{p}{q}$, where p and q are relatively prime positive integers. What is $p + q$?

- A 109
- B 201
- C 301
- D 3049
- E 33,601

Solution:

Setting $b = c = a$ gives $a \diamond 1 = a \diamond (a \diamond a) = (a \diamond a) \cdot a = a$. Then setting $c = b$ gives $a = a \diamond 1 = a \diamond (b \diamond b) = (a \diamond b) \cdot b$, so $a \diamond b = \frac{a}{b}$.

Therefore

$$\begin{aligned} 2016 \diamond (6 \diamond x) &= 2016 \diamond \frac{6}{x} \\ &= \frac{2016}{\frac{6}{x}} \\ &= 336x = 100, \end{aligned}$$

$$\text{so } x = \frac{100}{336} = \frac{25}{84} \text{ and } p + q = 25 + 84 = 109.$$

Thus, the correct answer is **A**.

21. A quadrilateral is inscribed in a circle of radius $200\sqrt{2}$. Three of the sides of this quadrilateral have length 200. What is the length of its fourth side?

A 200

B $200\sqrt{2}$

C $200\sqrt{3}$

D $300\sqrt{2}$

E 500

Solution:

Let θ be the central angle subtending a side of length 200, with radius $R = 200\sqrt{2}$. By the law of cosines on the isosceles triangle from the center,

$$\begin{aligned} 200^2 &= 2R^2(1 - \cos \theta) \\ &= 160000(1 - \cos \theta), \end{aligned}$$

$$\text{so } \cos \theta = \frac{3}{4}.$$

The three equal sides use three consecutive arcs of angle θ , so the fourth arc has angle $2\pi - 3\theta$. Its chord has the same length as a chord with central angle 3θ , and

$$\begin{aligned} \cos 3\theta &= 4 \cos^3 \theta - 3 \cos \theta \\ &= 4 \cdot \frac{27}{64} - \frac{9}{4} \\ &= -\frac{9}{16}. \end{aligned}$$

Its length squared is

$$\begin{aligned} & 2R^2(1 - \cos 3\theta) \\ &= 160000 \left(1 + \frac{9}{16}\right) \\ &= 160000 \cdot \frac{25}{16} \\ &= 250000, \end{aligned}$$

so the fourth side is 500.

Thus, the correct answer is **E**.

22. How many ordered triples (x, y, z) of positive integers satisfy $\text{lcm}(x, y) = 72$, $\text{lcm}(x, z) = 600$, and $\text{lcm}(y, z) = 900$?

A 15

B 16

C 24

D 27

E 64

Solution:

Because $\text{lcm}(x, y) = 2^3 \cdot 3^2$ and $\text{lcm}(x, z) = 2^3 \cdot 3 \cdot 5^2$, the factor 5^2 divides z while neither x nor y is divisible by 5. Also 3^2 divides y , while neither x nor z is divisible by 3^2 , and x must have the factor 2^3 .

Writing $x = 2^3 \cdot 3^j$, $y = 2^k \cdot 3^2$, and $z = 2^m \cdot 3^n \cdot 5^2$, the lcm conditions require $\max(j, n) = 1$ and $\max(k, m) = 2$. The first pair can be $(1, 0)$, $(0, 1)$, $(1, 1)$, and the second can be $(2, 0)$, $(2, 1)$, $(2, 2)$, $(0, 2)$, $(1, 2)$. Thus there are $3 \cdot 5 = 15$ ordered triples.

Thus, the correct answer is **A**.

23. Three numbers in the interval $[0, 1]$ are chosen independently and at random. What is the probability that the chosen numbers are the side lengths of a triangle with positive area?

A $\frac{1}{6}$

B $\frac{1}{3}$

C $\frac{1}{2}$

D $\frac{2}{3}$

E $\frac{5}{6}$

Solution:

The ordered triples (x, y, z) fill the unit cube of volume 1. They fail to form a triangle exactly when one value is at least the sum of the other two.

The region $z \geq x + y$ is a tetrahedron with vertices $(0, 0, 0)$, $(0, 0, 1)$, $(0, 1, 1)$, $(1, 0, 1)$ of volume $\frac{1}{6}$. The analogous regions $x \geq y + z$ and $y \geq x + z$ also have volume $\frac{1}{6}$ and have disjoint interiors. So the failure probability is $3 \cdot \frac{1}{6} = \frac{1}{2}$, and the triangle probability is $1 - \frac{1}{2} = \frac{1}{2}$.

Thus, the correct answer is **C**.

24. There is a smallest positive real number a such that there exists a positive real number b such that all the roots of the polynomial $x^3 - ax^2 + bx - a$ are real. In fact, for this value of a the value of b is unique. What is this value of b ?

A 8

B 9

C 10

D 11

E 12

Solution:

By Vieta's formulas, the real roots r, s, t satisfy $r + s + t = a$, $rs + st + tr = b$, and $rst = a$, so $r + s + t = rst$.

The positive product means either all three roots are positive or two are negative. In the latter case write the roots as $-u, -v, w$ with $u, v, w > 0$. The positive sum forces $w > u + v$, and then $b = uv - w(u + v) < 0$, a contradiction. Hence all three roots are positive.

By the AM-GM inequality, $27rst \leq (r + s + t)^3 = (rst)^3$, so $a = rst \geq 3\sqrt{3}$, with equality if and only if $r = s = t = \sqrt{3}$. At this smallest a ,

$$\begin{aligned} b &= rs + st + tr \\ &= 3r^2 = 3 \cdot 3 = 9. \end{aligned}$$

Thus, the correct answer is **B**.

25. Let k be a positive integer. Bernardo and Silvia take turns writing and erasing numbers on a blackboard as follows: Bernardo starts by writing the smallest perfect square with $k + 1$ digits. Every time Bernardo writes a number, Silvia erases the last k digits of it. Bernardo then writes the next perfect square, Silvia erases the last k digits of it, and this process continues until the last two numbers that remain on the board differ by at least 2. Let $f(k)$ be the smallest positive integer not written on the board. For example, if $k = 1$, then the numbers that Bernardo writes are 16, 25, 36, 49, and 64, and the numbers showing on the board after Silvia erases are 1, 2, 3, 4, and 6, and thus $f(1) = 5$. What is the sum of the digits of $f(2) + f(4) + f(6) + \cdots + f(2016)$?

A 7986

B 8002

C 8030

D 8048

E 8064

Solution:

Take $k = 2j$. The smallest perfect square with $k + 1$ digits is $10^k = (10^j)^2$, and after Silvia erases, the numbers shown are $\left\lfloor \frac{n^2}{10^k} \right\rfloor$ for $n = 10^j, 10^j + 1, \dots$

Put $M = 10^k$. A jump of at least 2 from n to $n + 1$ requires $(n + 1)^2 - n^2 = 2n + 1 > M$, so write $n = \frac{M}{2} + m$ with $m \geq 0$. The case $m = 0$ gives a jump of only 1, so the first larger jump has $m \geq 1$. Let $A = \left\lfloor \frac{n^2}{M} \right\rfloor$ and $B = \left\lfloor \frac{(n+1)^2}{M} \right\rfloor$. Because M is divisible by 4,

$$A = \frac{M}{4} + m + \left\lfloor \frac{m^2}{M} \right\rfloor,$$

$$B = \frac{M}{4} + m + 1 + \left\lfloor \frac{(m+1)^2}{M} \right\rfloor.$$

Therefore the first jump of at least 2 occurs at the first m for which $m^2 < M \leq (m + 1)^2$. Since $\sqrt{M} = 10^j$, this is $m = 10^j - 1$. The last displayed value before the gap is $\frac{M}{4} + 10^j - 1$, so the smallest missing integer is

$$f(2j) = \frac{10^{2j}}{4} + 10^j.$$

Summing over $j = 1, \dots, 1008$,

$$\begin{aligned} & \sum_{j=1}^{1008} f(2j) \\ &= 25 \sum_{j=0}^{1007} 10^{2j} \\ & \quad + 10 \sum_{j=0}^{1007} 10^j \\ &= \underbrace{2525 \dots 25}_{2016 \text{ digits}} \\ & \quad + \underbrace{111 \dots 10}_{1009 \text{ digits}}. \end{aligned}$$

There are no carries, so the digit sum is $1008 \cdot (2 + 5) + 1008 \cdot 1 = 1008 \cdot 8 = 8064$.

Thus, the correct answer is **E**.

Problems: <https://live.poshenloh.com/past-contests/amc12/2016A>



2016 AMC 12B Solutions

Typeset by: LIVE by Po-Shen Loh

<https://live.poshenloh.com/past-contests/amc12/2016B/solutions>



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1. What is the value of

$$\frac{2a^{-1} + \frac{a^{-1}}{2}}{a}$$

when $a = \frac{1}{2}$?

- ☐ A 1
- ☐ B 2
- ☐ C $\frac{5}{2}$
- ☒ D 10
- ☐ E 20

Solution:

With $a = \frac{1}{2}$, we have $a^{-1} = 2$. The numerator is $2 \cdot 2 + \frac{2}{2} = 4 + 1 = 5$, and dividing by $a = \frac{1}{2}$ gives $\frac{5}{\frac{1}{2}} = 10$.

Thus, the correct answer is **D**.

2. The harmonic mean of two numbers can be computed as twice their product divided by their sum. The harmonic mean of 1 and 2016 is closest to which integer?

A 2

B 45

C 504

D 1008

E 2015

Solution:

The harmonic mean is $\frac{2 \cdot 1 \cdot 2016}{1 + 2016} = \frac{4032}{2017}$. Since $\frac{2016}{2017}$ is very close to 1, this is just under 2, so the closest integer is 2.

Thus, the correct answer is **A**.

3. Let $x = -2016$. What is the value of

$$\left| \left| |x| - x \right| - |x| \right| - x?$$

A -2016

B 0

C 2016

D 4032

E 6048

Solution:

Since $x = -2016$, $|x| = 2016$. The innermost expression is $|x| - x = 2016 + 2016 = 4032$. Then $|4032| - |x| = 4032 - 2016 = 2016$, and the outer absolute value leaves 2016 . Finally subtracting x gives $2016 - (-2016) = 4032$.

Thus, the correct answer is **D**.

4. The ratio of the measures of two acute angles is $5 : 4$, and the complement of one of these two angles is twice as large as the complement of the other. What is the sum of the degree measures of the two angles?

A 75

B 90

C 135

D 150

E 270

Solution:

Let the angles be $\alpha < \beta$ with $\beta = \frac{5}{4}\alpha$. The larger complement belongs to the smaller angle, so $90 - \alpha = 2(90 - \beta) = 180 - \frac{5}{2}\alpha$. This gives $\frac{3}{2}\alpha = 90$, so $\alpha = 60^\circ$ and $\beta = 75^\circ$. The sum is 135° .

Thus, the correct answer is **C**.

5. The War of 1812 started with a declaration of war on Thursday, June 18, 1812. The peace treaty to end the war was signed 919 days later, on December 24, 1814. On what day of the week was the treaty signed?

- ☐ A Friday
- ☒ B Saturday
- ☐ C Sunday
- ☐ D Monday
- ☐ E Tuesday

Solution:

Because $919 = 7 \cdot 131 + 2$, the treaty was signed 131 full weeks plus 2 days after Thursday. Two days beyond Thursday is Saturday.

Thus, the correct answer is **B**.

6. All three vertices of $\triangle ABC$ lie on the parabola defined by $y = x^2$, with A at the origin and $\overline{BC}$ parallel to the x -axis. The area of the triangle is 64. What is the length of BC ?

A 4

B 6

C 8

D 10

E 16

Solution:

Let the vertex in the first quadrant be (x, x^2) . By symmetry the base is $BC = 2x$ and the height is x^2 , so $\frac{1}{2} \cdot 2x \cdot x^2 = x^3 = 64$. Thus $x = 4$ and $BC = 2x = 8$.

Thus, the correct answer is **C**.

7. Josh writes the numbers $1, 2, 3, \dots, 99, 100$. He marks out 1 , skips the next number (2), marks out 3 , and continues skipping and marking out the next number to the end of his list. Then he goes back to the start of his list, marks out the first remaining number (2), skips the next number (4), marks out 6 , skips 8 , marks out 10 , and so on to the end. Josh continues in this manner until only one number remains. What is that number?

A 13

B 32

C 56

D 64

E 96

Solution:

The first pass removes the odd numbers, leaving the multiples of 2 . The second pass removes $2, 6, 10, \dots$, leaving the multiples of 4 . In general, after the n th pass only the multiples of 2^n remain. The surviving number is the highest power of 2 not exceeding 100 , which is $2^6 = 64$.

Thus, the correct answer is **D**.

8. A thin piece of wood of uniform density in the shape of an equilateral triangle with side length 3 inches weighs 12 ounces. A second piece of the same type of wood, with the same thickness, also in the shape of an equilateral triangle, has side length 5 inches. Which of the following is closest to the weight, in ounces, of the second piece?

A 14.0

B 16.0

C 20.0

D 33.3

E 55.6

Solution:

Weight is proportional to area, and area scales with the square of the side length. The second side is $\frac{5}{3}$ times the first, so its weight is $12 \cdot \left(\frac{5}{3}\right)^2 = \frac{100}{3} \approx 33.3$ ounces.

Thus, the correct answer is **D**.

9. Carl decided to fence in his rectangular garden. He bought 20 fence posts, placed one on each of the four corners, and spaced out the rest evenly along the edges of the garden, leaving exactly 4 yards between neighboring posts. The longer side of his garden, including the corners, has twice as many posts as the shorter side, including the corners. What is the area, in square yards, of Carl's garden?

A 256

B 336

C 384

D 448

E 512

Solution:

Let the shorter side have x posts, so the longer side has $2x$. Counting all posts and subtracting the four corners counted twice, $2x + 2(2x) - 4 = 20$, giving $x = 4$. The shorter side has 4 posts, or $(4 - 1) \cdot 4 = 12$ yards, and the longer side has 8 posts, or $(8 - 1) \cdot 4 = 28$ yards. The area is $12 \cdot 28 = 336$.

Thus, the correct answer is **B**.

10. A quadrilateral has vertices $P(a, b)$, $Q(b, a)$, $R(-a, -b)$, and $S(-b, -a)$, where a and b are integers with $a > b > 0$. The area of $PQRS$ is 16. What is $a + b$?

- ☒ A 4
- ☐ B 5
- ☐ C 6
- ☐ D 12
- ☐ E 13

Solution:

The sides $\overline{PQ}$ and $\overline{RS}$ have slope -1 , and $\overline{QR}$ and $\overline{PS}$ have slope 1 , so $PQRS$ is a rectangle with sides $(a - b)\sqrt{2}$ and $(a + b)\sqrt{2}$. Its area is $2(a - b)(a + b) = 2(a^2 - b^2) = 16$, so $a^2 - b^2 = 8$. The only perfect squares differing by 8 are 9 and 1, giving $a = 3$, $b = 1$, and $a + b = 4$.

Thus, the correct answer is **A**.

11. How many squares whose sides are parallel to the axes and whose vertices have coordinates that are integers lie entirely within the region bounded by the line $y = \pi x$, the line $y = -0.1$, and the line $x = 5.1$?

A 30

B 41

C 45

D 50

E 57

Solution:

A square whose left edge is $x = k$ must fit below the lowest point of $y = \pi x$ over its width, namely $y = \pi k$. For side length 1, the possible left edges $k = 1, 2, 3, 4$ contribute $3 + 6 + 9 + 12 = 30$ squares. For side length 2, the left edges $k = 1, 2, 3$ contribute $2 + 5 + 8 = 15$, and for side length 3, the left edges $k = 1, 2$ contribute $1 + 4 = 5$. A square of side at least 4 cannot fit, so the total is $30 + 15 + 5 = 50$.

Thus, the correct answer is **D**.

12. All the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9 are written in a 3×3 array of squares, one number in each square, in such a way that if two numbers are consecutive then they occupy squares that share an edge. The numbers in the four corners add up to 18. What number is in the center?

- A 5
- B 6
- C 7**
- D 8
- E 9

Solution:

Color the grid like a checkerboard so the four corners and the center share one color. Since consecutive numbers occupy adjacent (opposite colored) squares, the numbers alternate parity along the chain, so the five same-colored cells contain the five odd numbers 1, 3, 5, 7, 9, which sum to 25. The four corners add to 18, so the center is $25 - 18 = 7$.

Thus, the correct answer is **C**.

13. Alice and Bob live 10 miles apart. One day Alice looks due north from her house and sees an airplane. At the same time Bob looks due west from his house and sees the same airplane. The angle of elevation of the airplane is 30° from Alice's position and 60° from Bob's position. Which of the following is closest to the airplane's altitude, in miles?

A 3.5

B 4

C 4.5

D 5

E 5.5

Solution:

Let the airplane be at C , directly above point D on the ground at altitude h . Triangles ACD and BCD are 30-60-90 right triangles, so $AD = \sqrt{3}h$ and $BD = \frac{h}{\sqrt{3}}$. Since Alice looks north and Bob looks west, $\angle ADB = 90^\circ$, so $AD^2 + BD^2 = AB^2 = 100$. Then $3h^2 + \frac{h^2}{3} = \frac{10h^2}{3} = 100$, giving $h = \sqrt{30} \approx 5.48$, closest to 5.5.

Thus, the correct answer is **E**.

14. The sum of an infinite geometric series is a positive number S , and the second term in the series is 1. What is the smallest possible value of S ?

A $\frac{1 + \sqrt{5}}{2}$

B 2

C $\sqrt{5}$

D 3

E 4

Solution:

Let r be the common ratio. Since the second term is 1, the first term is $\frac{1}{r}$, so $S = \frac{\frac{1}{r}}{1 - r} = \frac{1}{r - r^2}$. Convergence requires $|r| < 1$, and $S > 0$ then forces $0 < r < 1$. Therefore S is smallest when $r - r^2$ is largest. The parabola $r - r^2$ peaks at $r = \frac{1}{2}$, where it equals $\frac{1}{4}$, so the smallest value of S is $\frac{1}{\frac{1}{4}} = 4$.

Thus, the correct answer is **E**.

15. All the numbers 2, 3, 4, 5, 6, 7 are assigned to the six faces of a cube, one number to each face. For each of the eight vertices of the cube, a product of three numbers is computed, where the three numbers are the numbers assigned to the three faces that include that vertex. What is the greatest possible value of the sum of these eight products?

A 312

B 343

C 625

D 729

E 1680

Solution:

Pair the opposite faces as (a, b) , (c, d) , (e, f) . Each vertex product uses one face from each pair, so the sum of all eight products factors as $(a + b)(c + d)(e + f)$. The three factors have fixed total $2 + 3 + 4 + 5 + 6 + 7 = 27$, and a product with fixed sum is largest when the factors are equal, at 9 each. This balance is achievable with $(2, 7)$, $(3, 6)$, $(4, 5)$, giving $9 \cdot 9 \cdot 9 = 729$.

Thus, the correct answer is **D**.

16. In how many ways can 345 be written as the sum of an increasing sequence of two or more consecutive positive integers?

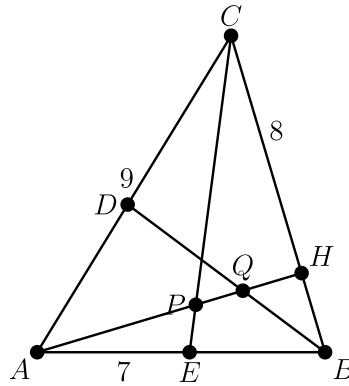
- ☐ A 1
- ☐ B 3
- ☐ C 5
- ☐ D 6
- ☒ E 7

Solution:

A sum of consecutive integers equals the count times the median. For an odd number of terms, the median is an integer divisor of 345, giving runs of 3 (median 115), 5 (median 69), 15 (median 23), and 23 (median 15) terms. For an even number of terms the median is a half-integer. The positive possibilities have lengths 2, 6, 10, with respective medians 172.5, 57.5, 34.5. Longer divisor-based runs would force a nonpositive first term. This gives $4 + 3 = 7$ ways.

Thus, the correct answer is **E**.

17. In $\triangle ABC$ shown in the figure, $AB = 7$, $BC = 8$, $CA = 9$, and $\overline{AH}$ is an altitude. Points D and E lie on sides $\overline{AC}$ and $\overline{AB}$, respectively, so that $\overline{BD}$ and $\overline{CE}$ are angle bisectors, intersecting $\overline{AH}$ at Q and P , respectively. What is PQ ?



- A 1
- B $\frac{5}{8}\sqrt{3}$
- C $\frac{4}{5}\sqrt{2}$
- D $\frac{8}{15}\sqrt{5}$**
- E $\frac{6}{5}$

Solution:

Let $x = BH$. Then $CH = 8 - x$, and from the two right triangles $AH^2 = 7^2 - x^2 = 9^2 - (8 - x)^2$. This gives $x = 2$ and $AH = \sqrt{45}$. By the angle bisector theorem in $\triangle ACH$, $\frac{AP}{PH} = \frac{CA}{CH} = \frac{9}{6}$, so $AP = \frac{3}{5}AH$. Similarly in $\triangle ABH$, $\frac{AQ}{QH} = \frac{BA}{BH} =$

$\frac{7}{2}$, so $AQ = \frac{7}{9}AH$. Then

$$\begin{aligned}PQ &= AQ - AP \\&= \left(\frac{7}{9} - \frac{3}{5}\right)AH \\&= \frac{8}{45}\sqrt{45} \\&= \frac{8}{15}\sqrt{5}.\end{aligned}$$

Thus, the correct answer is **D**.

18. What is the area of the region enclosed by the graph of the equation $x^2 + y^2 = |x| + |y|$?

A $\pi + \sqrt{2}$

B $\pi + 2$

C $\pi + 2\sqrt{2}$

D $2\pi + \sqrt{2}$

E $2\pi + 2\sqrt{2}$

Solution:

By symmetry, consider the first quadrant, where the equation is $x^2 + y^2 = x + y$, or $\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{2}$. This is a circle centered at $\left(\frac{1}{2}, \frac{1}{2}\right)$ passing through $(1, 0)$ and $(0, 1)$; since the center is the midpoint of that chord, the enclosed first-quadrant region is the right triangle with legs to $(1, 0)$ and $(0, 1)$ (area $\frac{1}{2}$) plus a semicircle of radius $\frac{\sqrt{2}}{2}$ (area $\frac{\pi}{4}$). Multiplying by 4 for all quadrants gives $4\left(\frac{1}{2} + \frac{\pi}{4}\right) = \pi + 2$.

Thus, the correct answer is **B**.

19. Tom, Dick, and Harry are playing a game. Starting at the same time, each of them flips a fair coin repeatedly until he gets his first head, at which point he stops. What is the probability that all three flip their coins the same number of times?

A $\frac{1}{8}$

B $\frac{1}{7}$

C $\frac{1}{6}$

D $\frac{1}{4}$

E $\frac{1}{3}$

Solution:

A player's first head comes on flip n with probability $\left(\frac{1}{2}\right)^n$. All three stopping on the same flip n has probability $\left(\left(\frac{1}{2}\right)^n\right)^3 = \left(\frac{1}{8}\right)^n$. Summing over $n \geq 1$, $\sum_{n=1}^{\infty} \left(\frac{1}{8}\right)^n =$

$$\frac{\frac{1}{8}}{1 - \frac{1}{8}} = \frac{1}{7}.$$

Thus, the correct answer is **B**.

20. A set of teams held a round-robin tournament in which every team played every other team exactly once. Every team won 10 games and lost 10 games; there were no ties. How many sets of three teams $\{A, B, C\}$ were there in which A beat B , B beat C , and C beat A ?

A 385

B 665

C 945

D 1140

E 1330

Solution:

Since each team won 10 and lost 10, there are 21 teams and $\binom{21}{3} = 1330$ triples. A triple is not cyclic exactly when one team beats both others. Choosing that team (21 ways) and 2 of the 10 teams it beat gives $21 \cdot \binom{10}{2} = 21 \cdot 45 = 945$ non-cyclic triples. Thus the cyclic triples number $1330 - 945 = 385$.

Thus, the correct answer is **A**.

21. Let $ABCD$ be a unit square. Let Q_1 be the midpoint of $\overline{CD}$. For $i = 1, 2, \dots$, let P_i be the intersection of $\overline{AQ_i}$ and $\overline{BD}$, and let Q_{i+1} be the foot of the perpendicular from P_i to $\overline{CD}$. What is

$$\sum_{i=1}^{\infty} \text{Area of } \triangle DQ_iP_i?$$

- A $\frac{1}{6}$
- B $\frac{1}{4}$**
- C $\frac{1}{3}$
- D $\frac{1}{2}$
- E 1

Solution:

Place $D = (0, 0)$, $C = (1, 0)$, $B = (1, 1)$, $A = (0, 1)$, and let $q_i = DQ_i$.

Intersecting line AQ_i with $\overline{BD}$ (the line $y = x$) gives P_i with both coordinates $\frac{q_i}{1 + q_i}$,

so $q_{i+1} = \frac{q_i}{1 + q_i}$. From $q_1 = \frac{1}{2}$ this yields $q_i = \frac{1}{i + 1}$. The base of $\triangle DQ_iP_i$ is

$DQ_i = \frac{1}{i + 1}$ and its height is the y -coordinate of P_i , which is $q_{i+1} = \frac{1}{i + 2}$. Then

$$\begin{aligned} \text{Area of } \triangle DQ_iP_i &= \frac{1}{2} \cdot \frac{1}{i + 1} \\ &\quad \cdot \frac{1}{i + 2} \\ &= \frac{1}{2} \left(\frac{1}{i + 1} - \frac{1}{i + 2} \right). \end{aligned}$$

Summing telescopes to $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$.

Thus, the correct answer is **B**.

22. For a certain positive integer n less than 1000, the decimal equivalent of $\frac{1}{n}$ is $0.\overline{abcdef}$, a repeating decimal of period 6, and the decimal equivalent of $\frac{1}{n+6}$ is $0.\overline{wxyz}$, a repeating decimal of period 4. In which interval does n lie?

- A [1, 200]
- B [201, 400]**
- C [401, 600]
- D [601, 800]
- E [801, 999]

Solution:

Period 6 requires $n \mid 10^6 - 1 = 3^3 \cdot 7 \cdot 11 \cdot 13 \cdot 37$. Period 4 requires $n + 6 \mid 10^4 - 1 = 3^2 \cdot 11 \cdot 101$ but $n + 6 \nmid 10^2 - 1 = 3^2 \cdot 11$ (else the period would be 1 or 2). Hence $101 \mid n + 6$. Since $n + 6$ also divides $3^2 \cdot 11 \cdot 101$ and is less than 1006, the only possibilities are $n + 6 = 101, 303, 909$, giving $n = 95, 297, 903$. Only $297 = 3^3 \cdot 11$ divides $10^6 - 1$, so $n = 297$.

Finally, $10^6 \equiv 1 \pmod{297}$, while $10^2 \not\equiv 1$ and $10^3 \not\equiv 1 \pmod{297}$, so its period is exactly 6. Also 303 divides $10^4 - 1$ but not $10^2 - 1$, so the period of $\frac{1}{303}$ is exactly 4. Thus $n = 297$ lies in $[201, 400]$.

Thus, the correct answer is **B**.

23. What is the volume of the region in three-dimensional space defined by the inequalities $|x| + |y| + |z| \leq 1$ and $|x| + |y| + |z - 1| \leq 1$?

A $\frac{1}{6}$

B $\frac{1}{3}$

C $\frac{1}{2}$

D $\frac{2}{3}$

E 1

Solution:

The region $|x| + |y| + |z| \leq 1$ is a regular octahedron with vertices at $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, $(0, 0, \pm 1)$, whose volume is $2 \cdot \frac{1}{3} \cdot (\sqrt{2})^2 \cdot 1 = \frac{4}{3}$. The second region is the same octahedron shifted up by 1. Their intersection is bounded by another regular octahedron with diagonals of length 1, half the linear dimensions of the first, so its volume is $\left(\frac{1}{2}\right)^3 \cdot \frac{4}{3} = \frac{1}{6}$.

Thus, the correct answer is **A**.

24. There are exactly 77,000 ordered quadruples (a, b, c, d) such that $\gcd(a, b, c, d) = 77$ and $\text{lcm}(a, b, c, d) = n$. What is the smallest possible value of n ?

A 13,860

B 20,790

C 21,560

D 27,720

E 41,580

Solution:

Writing each entry as 77 times a reduced value, we need $\gcd = 1$ and $\text{lcm} = m = \frac{n}{77}$. For each prime p dividing m with maximum exponent M , the number of valid exponent quadruples is $(M + 1)^4 - 2M^4 + (M - 1)^4 = 2(6M^2 + 1)$. The total over all primes must equal $77,000 = 2^3 \cdot 5^3 \cdot 7 \cdot 11$. Since $2(6M^2 + 1)$ equals 14, 50, and 110 for $M = 1, 2, 3$, and $14 \cdot 50 \cdot 110 = 77,000$, the exponents 1, 2, 3 give one candidate factorization.

Every prime contributes exactly one factor of 2, so exactly three primes divide m . Their odd factors $6M^2 + 1$ must divide $\frac{77,000}{2^3} = 9625$. Checking the divisors gives $M = 1, 2, 3, 8$, with odd factors 7, 25, 55, 385. The choice $M = 8$ leaves only 25 for the product of the other two odd factors, but each is at least 7, so this is impossible. Therefore the maximum exponents are exactly 1, 2, 3. To minimize $m = \frac{n}{77}$, assign the largest exponent to the smallest prime: $m = 2^3 \cdot 3^2 \cdot 5 = 360$, so $n = 77 \cdot 360 = 27,720$.

Thus, the correct answer is **D**.

25. The sequence (a_n) is defined recursively by $a_0 = 1$, $a_1 = \sqrt[19]{2}$, and $a_n = a_{n-1}a_{n-2}^2$ for $n \geq 2$. What is the smallest positive integer k such that the product $a_1a_2 \cdots a_k$ is an integer?

A 17

B 18

C 19

D 20

E 21

Solution:

Write $a_n = 2^{\frac{b_n}{19}}$. The recursion becomes $b_0 = 0$, $b_1 = 1$, $b_n = b_{n-1} + 2b_{n-2}$, solved by $b_n = \frac{1}{3}(2^n - (-1)^n)$. The product $a_1 \cdots a_k$ is an integer exactly when $19 \mid b_1 + \cdots + b_k$. Summing the formula for b_n gives $b_1 + \cdots + b_k = \frac{2^{k+1} - 1}{3}$ when k is odd, and $b_1 + \cdots + b_k = \frac{2^{k+1} - 2}{3}$ when k is even.

The order of 2 modulo 19 is 18 because $2^9 \equiv -1 \pmod{19}$ and $2^6 \not\equiv 1 \pmod{19}$. For odd k , divisibility therefore requires $18 \mid k + 1$, first occurring at $k = 17$. For even k , it requires $18 \mid k$, first occurring at $k = 18$. Hence the smallest positive k is 17.

Thus, the correct answer is **A**.

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