2022 AMC 10B Solutions

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1. Define $x \diamondsuit y$ to be $|x-y|$ for all real numbers x and y. What is the value of

$$(1 \diamondsuit (2 \diamondsuit 3)) - ((1 \diamondsuit 2) \diamondsuit 3)?$$

A -2

B -1

C 0

D 1

E 2

Solution:

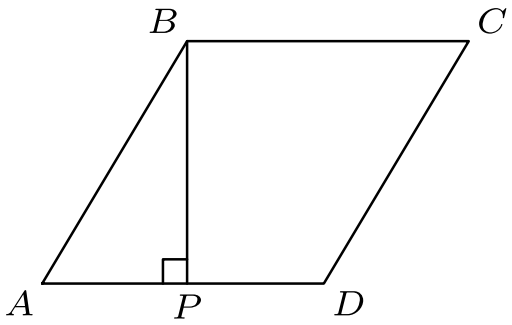
Working from the innermost operations outward gives

$$1 \diamondsuit (2 \diamondsuit 3) = |1 - |2-3|| = 0,$$
$$(1 \diamondsuit 2) \diamondsuit 3 = ||1-2| - 3| = 2.$$

Therefore, the given difference is $0 - 2 = -2.$

Thus, the answer is **A** .

2. In rhombus $ABCD$, point P lies on segment $\overline{AD}$ so that $\overline{BP} \perp \overline{AD}$, $AP = 3$, and $PD = 2$. What is the area of $ABCD$?



A $3\sqrt{5}$

B 10

C $6\sqrt{5}$

D 20

E 25

Solution:

Since $ABCD$ is a rhombus,

$$AB = AD = AP + PD = 5.$$

Right triangle ABP then gives

$$\begin{aligned} BP &= \sqrt{AB^2 - AP^2} \\ &= \sqrt{25 - 9} = 4. \end{aligned}$$

Thus the rhombus has base $AD = 5$ and height $BP = 4$, so its area is $5 \cdot 4 = 20$.

Thus, the answer is **D** .

3. How many three-digit positive integers have an odd number of even digits?

A 150

B 250

C 350

D 450

E 550

Solution:

There are $9 \cdot 10 = 90$ choices for the hundreds and tens digits. Once those two digits are fixed, the units digit must have whichever parity makes the total number of even digits odd. There are always 5 digits of the required parity (including 0 among the even digits).

Therefore, the number of integers is $90 \cdot 5 = 450.$

Thus, the answer is **D** .

4. A donkey suffers an attack of hiccups and the first hiccup happens at $4:00$ one afternoon. Suppose that the donkey hiccups regularly every 5 seconds. At what time does the donkey's 700th hiccup occur?

A 15 seconds after $4:58$

B 20 seconds after $4:58$

C 25 seconds after $4:58$

D 30 seconds after $4:58$

E 35 seconds after $4:58$

Solution:

Since we want to look at the 700th hiccup, we need to look at time that is 699 hiccups after the first one.

This would be $699 \cdot 5 = 3495$ seconds. Note that $3495 = 60 \cdot 58 + 15$, so the time would be 58 minutes and 15 seconds after the first hiccup. This would therefore be $4:58$ and 15 seconds.

Thus, the answer is **A**.

5. What is the value of

$$\frac{\left(1+\frac{1}{3}\right)\left(1+\frac{1}{5}\right)\left(1+\frac{1}{7}\right)}{\sqrt{\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{5^2}\right)\left(1-\frac{1}{7^2}\right)}}?$$

A $\sqrt{3}$

B 2

C $\sqrt{15}$

D 4

E $\sqrt{105}$

Solution:

Let the given positive expression be E. Squaring it and using $1-\frac{1}{p^2}=(1-\frac{1}{p})(1+\frac{1}{p})$ gives

$$\begin{aligned} E^2 &= \prod_{p\in\{3,5,7\}} \frac{1+\frac{1}{p}}{1-\frac{1}{p}} \\ &= \frac{4}{2}\cdot\frac{6}{4}\cdot\frac{8}{6} = 4. \end{aligned}$$

Hence $E=2$.

Thus, the answer is **B**.

6. How many of the first ten numbers of the sequence

$$121, 11211, 1112111, \ldots$$

are prime numbers?

A 0

B 1

C 2

D 3

E 4

Solution:

We claim that none of these numbers can ever be prime.

We prove this claim by noticing that the nth number is

$$\sum_{k=0}^{2n} 10^k + 10^n = \sum_{k=0}^{n} 10^k +$$

$$\sum_{k=n}^{2n} 10^k = \sum_{k=0}^{n} 10^k + \sum_{k=0}^{n} 10^k \cdot 10^n$$

$$= (10^n + 1)(\sum_{k=0}^{n} 10^k).$$

This shows that the number can be written as the product of two numbers greater than 1, so there are no primes.

Thus, the answer is **A** .

7. For how many values of the constant k will the polynomial $x^2 + kx + 36$ have two distinct integer roots?

A 6

B 8

C 9

D 14

E 16

Solution:

Let the roots be r, s. Expanding $(x - r)(x - s)$ gives $x^2 - (r + s)x + rs$.
Comparing coefficients with the given polynomial yields $rs = 36$ and $r + s = -k$.

Therefore, we need r and s distinct such that $rs = 36$. All the possible factor pairs are

$$\pm\{1, 36\}, \pm\{2, 18\}, \pm\{3, 12\}$$

and

$$\pm\{4, 9\}.$$

Each of these unordered pairs produces a unique value for k, so there are 8 possible values for k.

Thus, **B** is the correct answer.

8. Consider the following 100 sets of 10 elements each:

$$\{1, 2, 3, \ldots, 10\},$$
$$\{11, 12, 13, \ldots, 20\},$$
$$\{21, 22, 23, \ldots, 30\},$$
$$\vdots$$
$$\{991, 992, 993, \ldots, 1000\}.$$

How many of these sets contain exactly two multiples of 7?

A 40

B 42

C 43

D 49

E 50

Solution:

A block of ten consecutive integers contains exactly two multiples of 7 precisely when its first multiple of 7 is in one of the first three positions. Thus that multiple must have units digit $1, 2,$ or 3.

The multiples of 7 with those units digits are, respectively,

$$21 + 70j,$$
$$42 + 70j,$$
$$63 + 70j.$$

For each expression, $j = 0, 1, \ldots, 13$ gives a value at most 1000, so each class contributes 14 blocks. Therefore, the total is $3 \cdot 14 = 42$.

Thus, the answer is **B** .

9. The sum

$$\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\cdots+\frac{2021}{2022!}$$

can be expressed as $a-\frac{1}{b!}$, where a and b are positive integers. What is $a+b$?

A 2020

B 2021

C 2022

D 2023

E 2024

Solution:

Each term telescopes because

$$\frac{k}{(k+1)!}=\frac{1}{k!}-\frac{1}{(k+1)!}.$$

Summing makes every intermediate factorial reciprocal cancel, leaving

$$1-\frac{1}{2022!}.$$

Hence $a=1, b=2022$, and $a+b=2023.$

Thus, our answer is **D** .

10. Camila writes down five positive integers. The unique mode of these integers is 2 greater than their median, and the median is 2 greater than their arithmetic mean. What is the least possible value for the mode?

A 5

B 7

C 9

D 11

E 13

Solution:

Let the integers in increasing order be a, b, c, d, e. The median is c, and the unique mode is $c+2$.

Because the mode is larger than the median and is unique, the last two entries must both be $c+2$, so the list is $a, b, c, c+2, c+2$.

The mean is $c-2$, so

$$\frac{a+b+c+(c+2)+(c+2)}{5} = c-2.$$

Hence $a+b+3c+4 = 5c-10$, so $a+b = 2c-14$.

To keep the mode unique, a and b must be distinct positive integers, both less than c. Since $a+b = 2c-14$ is even, the smallest such sum is $1+3=4$, so $2c-14 \geq 4$, giving $c \geq 9$.

The smallest possible mode is therefore $c+2 = 11$, and it is attainable with $1, 3, 9, 11, 11$.

Thus, the answer is **D** .

11. All the high schools in a large school district are involved in a fundraiser selling T-shirts. Which of the choices below is logically equivalent to the statement "No school bigger than Euclid HS sold more T-shirts than Euclid HS"?

A All schools smaller than Euclid HS sold fewer T-shirts than Euclid HS.

B No school that sold more T-shirts than Euclid HS is bigger than Euclid HS.

C All schools bigger than Euclid HS sold fewer T-shirts than Euclid HS.

D All schools that sold fewer T-shirts than Euclid HS are smaller than Euclid HS.

E All schools smaller than Euclid HS sold more T-shirts than Euclid HS.

Solution:

The statement says: if a school is bigger than Euclid HS, then it did not sell more T-shirts than Euclid HS. Its contrapositive is: if a school sold more T-shirts than Euclid HS, then it is not bigger than Euclid HS. This is exactly choice **B** . Choice C is stronger than the original statement because it rules out a bigger school selling the same number of T-shirts.

Thus, the answer is **B** .

12. A pair of fair 6-sided dice is rolled n times. What is the least value of n such that the probability that the sum of the numbers face up on a roll equals 7 at least once is greater than $\frac{1}{2}$?

A 2

B 3

C 4

D 5

E 6

Solution:

To compute this, we can also find the least n such that the probability of not rolling a 7 is less than $\frac{1}{2}$. Each roll has an independent probability of $\frac{1}{6}$ of getting 7, so it has a $\frac{5}{6}$ probability of not landing on 7.

Thus, the probability of none of the rolls being 7 is $\left(\frac{5}{6}\right)^n$. We must find the least n such that $\left(\frac{5}{6}\right)^n < \frac{1}{2}$.

If $n = 3$, then the probability is $\frac{125}{216}$, which is greater than $\frac{1}{2}$.

If $n = 4$, then the probability is $\frac{625}{1296}$, which is less than $\frac{1}{2}$. This makes the answer 4.

Thus, the answer is **C**.

13. The positive difference between a pair of primes is equal to 2, and the positive difference between the cubes of the two primes is 31106. What is the sum of the digits of the least prime that is greater than those two primes?

A 8

B 10

C 11

D 13

E 16

Solution:

Since the primes are 2 away from each other, we can make them equal to $m-1, m+1$, where m is their average.

Then,

$$(m+1)^3-(m-1)^3=31106,$$

making

$$m^3+3m^2+3m+1$$
$$-(m^3-3m^2+3m-1)$$
$$=6m^2+2=31106.$$

Therefore, $m^2=5184$, so $m=72$.

The primes are therefore $71, 73$. The least prime greater than both of those is 79, and its digit sum is 16.

Thus, the answer is **E**.

14. Suppose that S is a subset of $\{1, 2, 3, \cdots, 25\}$ such that the sum of any two (not necessarily distinct) elements of S is never an element of S. What is the maximum number of elements S may contain?

A 12

B 13

C 14

D 15

E 16

Solution:

The set $S = \{13, 14 \cdots, 25\}$ has 13 elements, and every pair has sum greater than 25, so this size is attainable.

Conversely, let m be the maximum element of S. For every element of S satisfying $i < m$, the number i and the number $m - i$ cannot both belong to S.

Thus, among the numbers below m, at most one number can be chosen from each pair with sum m; if m is even, the middle number cannot be chosen either. Hence at most $\lfloor \frac{m-1}{2} \rfloor$ elements lie below m, and including m gives at most $\lfloor \frac{m-1}{2} \rfloor + 1$ elements.

The maximum value of this has $m = 25$, yielding 13.

Thus, the answer is **B** .

15. Let S_n be the sum of the first n terms of an arithmetic sequence that has a common difference of 2. The quotient $\dfrac{S_{3n}}{S_n}$ does not depend on n. What is S_{20}?

A 340

B 360

C 380

D 400

E 420

Solution:

Write the nth term as $a+2n$, so the term before the first term is a. Then

$$\begin{aligned} S_n &= \sum_{i=1}^{n}(a+2i) \\ &= n(a+n+1). \end{aligned}$$

Hence

$$\begin{aligned} \frac{S_{3n}}{S_n} &= \frac{3(a+3n+1)}{a+n+1} \\ &= 9 - \frac{6(a+1)}{a+n+1}. \end{aligned}$$

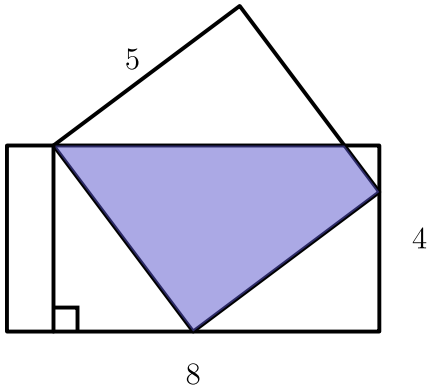
For this expression to be independent of n, its numerator $6(a+1)$ in the final fraction must be 0. Thus $a=-1$.

Therefore,

$$S_{20} = 20(-1+20+1) = 400.$$

Thus, the answer is **D** .

16. The diagram below shows a rectangle with side lengths 4 and 8 and a square with side length 5. Three vertices of the square lie on three different sides of the rectangle, as shown. What is the area of the region inside both the square and the rectangle?



A $15\frac{1}{8}$

B $15\frac{3}{8}$

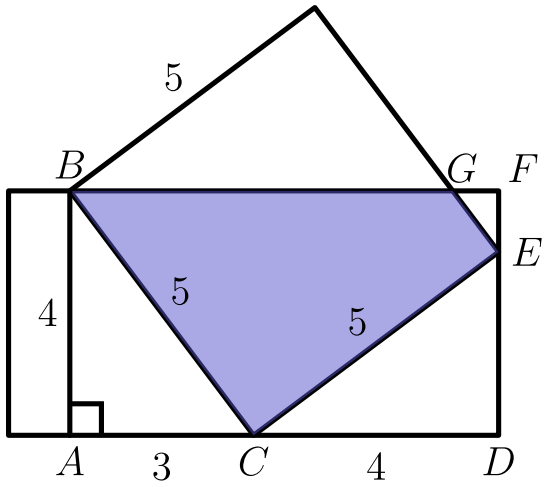
C $15\frac{1}{2}$

D $15\frac{5}{8}$

E $15\frac{7}{8}$

Solution:

Label the points as shown:



Because $AB = 4$ and the square side $BC = 5$, right triangle ABC gives $AC = 3$. Also $BC \perp CE$ and A, C, D are collinear, so $\angle ABC = \angle DCE$. The right triangles ABC and CDE have equal hypotenuses $BC = CE = 5$, so they are congruent. Thus $CD = 4$, $DE = 3$, and $EF = 4 - DE = 1$.

Right triangles EFG and CDE are similar, so

$$\frac{EG}{EF} = \frac{EC}{CD} = \frac{5}{4}.$$

Hence $EG = \frac{5}{4}$. The shaded region $BCEG$ is a trapezoid whose parallel sides are $BC = 5$ and $EG = \frac{5}{4}$, and whose height is the perpendicular side $CE = 5$. Its area is

$$\frac{1}{2}\left(5 + \frac{5}{4}\right)5 = \frac{125}{8} = 15\frac{5}{8}.$$

Thus, the answer is **D** .

17. One of the following numbers is not divisible by any prime number less than 10. Which is it?

A $2^{606}-1$

B $2^{606}+1$

C $2^{607}-1$

D $2^{607}+1$

E $2^{607}+3^{607}$

Solution:

Use the fact that $a^n - b^n$ is divisible by $a-b$.

Choice A is $2^{606}-1 = 4^{303}-1$, which is divisible by $4-1=3$.

Choice B is $2^{606}+1 = 4^{303}-(-1)^{303}$, which is divisible by $4-(-1)=5$.

Choice D is $2^{607}+1$. Since $2^{606}-1$ is divisible by 3, multiplying by 2 gives $2^{607}-2$ divisible by 3, so $2^{607}+1$ is divisible by 3.

Choice E is $3^{607}+2^{607} = 3^{607}-(-2)^{607}$, which is divisible by $3-(-2)=5$.

For choice C, $2^{607}-1$ is odd. Also $2^{607} \equiv 2 \pmod 3$, $2^{607} \equiv 3 \pmod 5$, and $2^{607} \equiv 2 \pmod 7$, so $2^{607}-1$ is not divisible by 3, 5, or 7.

Thus, our answer is **C**.

18. Consider systems of three linear equations with unknowns x, y, and z,

$$\begin{cases} a_1x + b_1y + c_1z &= 0 \\ a_2x + b_2y + c_2z &= 0 \\ a_3x + b_3y + c_3z &= 0 \end{cases}$$

where each of the coefficients is either 0 or 1 and the system has a solution other than $x = y = z = 0$. For example, one such system is

$$\begin{cases} 1x + 1y + 0z &= 0 \\ 0x + 1y + 1z &= 0 \\ 0x + 0y + 0z &= 0 \end{cases}$$

with a nonzero solution of $(x, y, z) = (1, -1, 1)$. How many such systems of equations are there? (The equations in a system need not be distinct, and two systems containing the same equations in a different order are considered different.)

A 302

B 338

C 340

D 343

E 344

Solution:

There are $2^9 = 512$ ordered binary coefficient matrices. A homogeneous system has only the zero solution exactly when its three row vectors are linearly independent, so we count those matrices and subtract.

An independent matrix must have three distinct nonzero rows. There are

$$7 \cdot 6 \cdot 5 = 210$$

ordered choices of such rows. Among three distinct nonzero binary vectors, dependence occurs exactly when one is the ordinary sum of the other two; the two summands must have disjoint nonempty supports.

If the sum has support of size 2, choose its two coordinates in 3 ways; its summands are the two corresponding unit vectors. If the sum has support of size 3, choose which one coordinate forms one summand in 3 ways, with the other two coordinates forming the other summand. Thus there are $3 + 3 = 6$ unordered dependent triples, each with $3! = 6$ row orders.

Hence the number of independent matrices is $210 - 6 \cdot 6 = 174$. The desired number of singular matrices, and therefore of systems with a nonzero solution, is

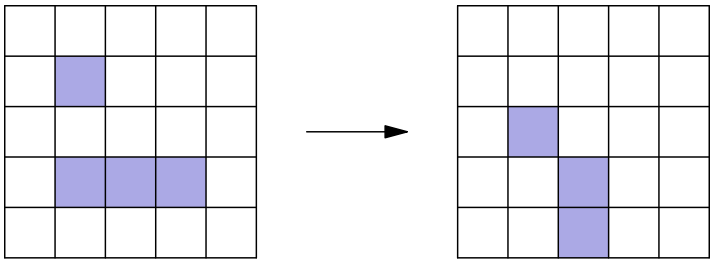
$$512 - 174 = 338.$$

Thus, the answer is **B** .

19. Each square in a 5×5 grid is either filled or empty, and has up to eight adjacent neighboring squares, where neighboring squares share either a side or a corner. The grid is transformed by the following rules:

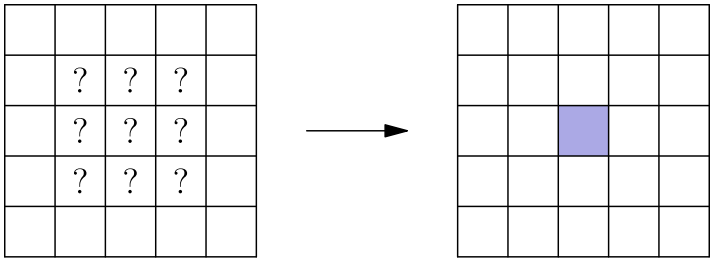
- Any filled square with two or three filled neighbors remains filled.
- Any empty square with exactly three filled neighbors becomes a filled square.
- All other squares remain empty or become empty.

A sample transformation is shown in the figure below.



Initial Transformed

Suppose the 5×5 grid has a border of empty squares surrounding a 3×3 subgrid. How many initial configurations will lead to a transformed grid consisting of a single filled square in the center after a single transformation? (Rotations and reflections of the same configuration are considered different.)



Initial Transformed

A 14

B 18

C 22

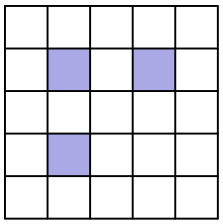
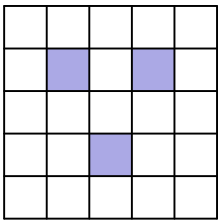
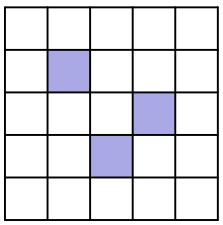
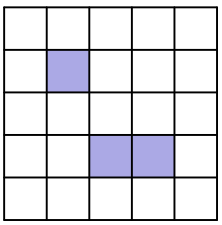
D 26

E 30

Solution:

First suppose the center is initially filled. It must have exactly 2 or 3 filled neighbors to survive. Every such neighbor already touches the center, so to disappear it cannot touch any other filled neighbor. Checking these pairwise nonadjacent positions, the only choices that do not also give some empty square exactly 3 filled neighbors are two opposite corners. There are 2 such configurations.

Now suppose the center is initially empty. Exactly 3 of its eight neighbors must be filled. Each of those three must disappear, so none may be adjacent to both of the others. Also, no empty square besides the center may be adjacent to all three. Applying these two tests gives the following four representative patterns:

Each of the first three patterns has 4 distinct rotations. The last has 4 rotations and their 4 reflected images, for 8 configurations. Thus the center-empty case contributes $4 + 4 + 4 + 8 = 20$, and the total is $20 + 2 = 22$.

Thus, the answer is **C** .

20. Let $ABCD$ be a rhombus with $\angle ADC = 46^\circ$. Let E be the midpoint of $\overline{CD}$, and let F be the point on $\overline{BE}$ such that $\overline{AF}$ is perpendicular to $\overline{BE}$. What is the degree measure of $\angle BFC$?

A 110

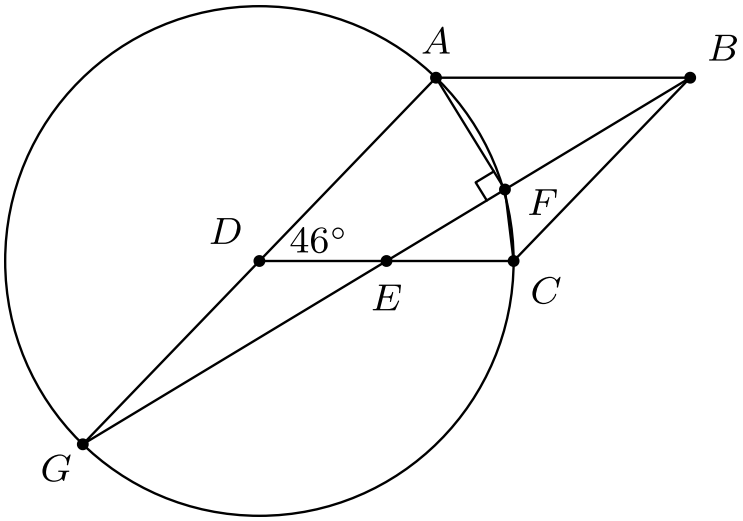
B 111

C 112

D 113

E 114

Solution:



Extend $\overline{BE}$ to meet line AD at G. Because $AD \parallel BC$, we have $\angle GDE = \angle ECB$, and $\angle GED = \angle BEC$ are vertical angles. Also $DE = EC$, so $\triangle GDE \cong \triangle BCE$. Hence $DG = BC = AD$.

Thus D is the midpoint of $\overline{AG}$. The circle centered at D through A also passes through C and G. Since $AF \perp FG$, Thales' theorem places F on this circle as well.

Because DG is opposite DA,

$$\begin{aligned} \angle GDC &= 180^\circ - \angle ADC \\ &= 134^\circ. \end{aligned}$$

The inscribed angle $\angle GFC$ subtending arc GC is therefore 67°. Finally, B, F, G are

collinear, so

$$\angle BFC = 180^\circ - 67^\circ = 113^\circ.$$

Thus, the answer is **D** .

21. Let $P(x)$ be a polynomial with rational coefficients such that when $P(x)$ is divided by the polynomial $x^2 + x + 1$, the remainder is $x + 2$, and when $P(x)$ is divided by the polynomial $x^2 + 1$, the remainder is $2x + 1$. There is a unique polynomial of least degree with these two properties. What is the sum of the squares of the coefficients of that polynomial?

A 10

B 13

C 19

D 20

E 23

Solution:

The first remainder condition gives

$$P(x) = (x^2 + x + 1)Q(x) + x + 2.$$

for some polynomial Q. Modulo $x^2 + 1$, we have $x^2 \equiv -1$, so

$$P(x) \equiv xQ(x) + x + 2.$$

If $Q(x) = c$ is constant, this remainder is $(c + 1)x + 2$, which cannot equal $2x + 1$. Thus Q must have degree at least 1.

Now let $Q(x) = ax + b$. Reducing modulo $x^2 + 1$ gives

$$P(x) \equiv (b + 1)x + (2 - a).$$

Matching this with $2x + 1$ yields $a = b = 1$. This constructs a degree-3 polynomial, and the failed constant case proves that degree 3 is minimal.

Therefore,

$$\begin{aligned} P(x) &= (x+1)(x^2+x+1) \\ &\quad + x + 2 \\ &= x^3 + 2x^2 + 3x + 3. \end{aligned}$$

The sum of the squares of its coefficients is $1^2 + 2^2 + 3^2 + 3^2 = 23.$

Thus, the answer is **E**.

22. Let S be the set of circles in the coordinate plane that are tangent to each of the three circles with equations

$$x^2 + y^2 = 4, \qquad x^2 + y^2 = 64,$$
$$\text{and} \qquad (x-5)^2 + y^2 = 3.$$

What is the sum of the areas of all circles in S?

A 48π

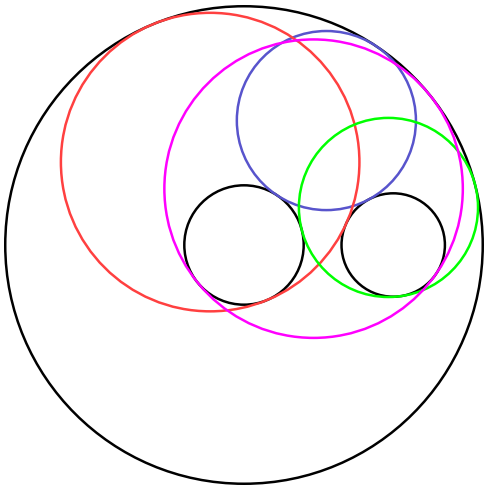
B 68π

C 96π

D 102π

E 136π

Solution:



Call the concentric circles of radii 2 and 8 the inner and outer circles. Let a desired circle have radius r and let its center be distance d from the origin. It must be internally tangent to the outer circle, so $d + r = 8$.

If it is externally tangent to the inner circle, then $d - r = 2$, giving $(r, d) = (3, 5)$. If it contains the inner circle, then $r - d = 2$, giving $(r, d) = (5, 3)$. Thus every desired circle has radius 3 or 5.

The third given circle has radius $\sqrt{3}$ and center $(5, 0)$. For either value of r, a desired circle may be internally or externally tangent to it, so the distance between their centers is $r - \sqrt{3}$ or $r + \sqrt{3}$. In all four cases, if this distance is q, then $|d - q| < 5 < d + q$, so the circle of possible centers intersects the circle of radius d about the origin in two points, symmetric across the x-axis, as shown. Hence there are 4 desired circles of each radius.

The total area is therefore

$$4(5^2\pi + 3^2\pi) = 136\pi.$$

Thus, **E** is the correct answer.

23. Ant Amelia starts on the number line at 0 and crawls in the following manner. For $n = 1, 2, 3$; Amelia chooses a time duration t_n and an increment x_n independently and uniformly at random from the interval $(0, 1)$. During the nth step of the process, Amelia moves x_n units in the positive direction, using up t_n minutes. If the total elapsed time has exceeded 1 minute during the nth step, she stops at the end of that step; otherwise, she continues with the next step, taking at most 3 steps in all. What is the probability that Amelia's position when she stops will be greater than 1?

A $\frac{1}{3}$

B $\frac{1}{2}$

C $\frac{2}{3}$

D $\frac{3}{4}$

E $\frac{5}{6}$

Solution:

The stopping time depends only on the time variables, while the final position depends only on the distance variables, so the corresponding probabilities multiply.

For two independent numbers in $(0, 1)$, the probability that their sum is less than 1 is the area of a right triangle, namely $\frac{1}{2}$.

For three independent numbers in $(0, 1)$, the probability that their sum is less than 1 is the volume of a tetrahedron with side intercepts 1, namely $\frac{1}{6}$.

If $t_1 + t_2 > 1$, Amelia stops after two steps. This has probability $\frac{1}{2}$, and independently $x_1 + x_2 > 1$ has probability $\frac{1}{2}$, contributing $\frac{1}{4}$.

If $t_1 + t_2 < 1$, Amelia takes the third step. This has probability $\frac{1}{2}$, and independently $x_1 + x_2 + x_3 > 1$ has probability $1 - \frac{1}{6} = \frac{5}{6}$, contributing $\frac{5}{12}$.

The total probability is $\frac{1}{4} + \frac{5}{12} = \frac{2}{3}$.

Thus, the answer is **C**.

24. Consider functions f that satisfy

$$|f(x) - f(y)| \le \frac{1}{2}|x - y|$$

for all real numbers x and y. Of all such functions that also satisfy the equation $f(300) = f(900)$, what is the greatest possible value of

$$f(f(800)) - f(f(400))?$$

A 25

B 50

C 100

D 150

E 200

Solution:

Applying the contraction inequality twice gives

$$\begin{aligned} &|f(f(400)) - f(f(300))| \\ &\quad \le \frac{1}{2}|f(400) - f(300)| \\ &\quad \le 25, \end{aligned}$$

and similarly

$$|f(f(800)) - f(f(900))| \le 25.$$

Set $M = f(f(300)) = f(f(900))$. The triangle inequality now yields

$$\begin{aligned} &|f(f(800)) - f(f(400))| \\ &\quad \le |f(f(800)) - M| \\ &\qquad + |M - f(f(400))| \\ &\quad \le 50. \end{aligned}$$

To attain the bound, define f by linear interpolation through the points

$$(300, 600), (400, 550), (550, 575),$$
$$(650, 625), (800, 650), (900, 600).$$

and set $f(x) = 600$ for $x \leq 300$ or $x \geq 900$. Every segment has slope with absolute value at most $\frac{1}{2}$, so the contraction condition holds. In particular, $f(300) = f(900) = 600$, $f(400) = 550$, and $f(800) = 650$. Hence $f(f(400)) = f(550) = 575$, while $f(f(800)) = f(650) = 625$, giving the difference 50.

Thus, the answer is **B** .

25. Let $x_0, x_1, x_2, \ldots$ be a sequence of numbers, where each x_k is either 0 or 1. For each positive integer n, define

$$S_n = \sum_{k=0}^{n-1} x_k 2^k$$

Suppose $7S_n \equiv 1 \pmod{2^n}$ for all $n \geq 1$. What is the value of the sum

$$x_{2019} + 2x_{2020} +$$

$$4x_{2021} + 8x_{2022}?$$

A 6

B 7

C 12

D 14

E 15

Solution:

The desired sum is

$$\frac{S_{2023} - S_{2019}}{2^{2019}}.$$

Also, $0 \leq S_n < 2^n$.

Therefore, for a unique $m_n \in \{0, 1, \ldots, 6\}$,

$$7S_n = m_n 2^n + 1.$$

Reducing modulo 7 gives $m_n 2^n \equiv -1 \pmod{7}$.

Since $2^3 \equiv 1 \pmod{7}$ and $2019 \equiv 0 \pmod{3}$, we get $m_{2019} = 6$. Since $2023 \equiv 1 \pmod{3}$, we have $2m_{2023} \equiv -1 \pmod{7}$, so $m_{2023} = 3$. Hence

$$S_{2019} = \frac{6 \cdot 2^{2019} + 1}{7},$$
$$S_{2023} = \frac{3 \cdot 2^{2023} + 1}{7}.$$

Finally,

$$\frac{S_{2023} - S_{2019}}{2^{2019}} = \frac{3 \cdot 2^4 - 6}{7} = 6.$$

Thus, the correct answer is **A** .

Problems: https://live.poshenloh.com/past-contests/amc10/2022B