

2021 AMC 10A Spring Solutions

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1. What is the value of

$$(2^2 - 2) - (3^2 - 3) + (4^2 - 4)?$$

- ☐ A 1
- ☐ B 2
- ☐ C 5
- ☒ D 8
- ☐ E 12

Solution:

$$\begin{aligned}(2^2 - 2) - (3^2 - 3) + (4^2 - 4) \\&= 2 - 6 + 12 \\&= 8.\end{aligned}$$

Thus, **D** is the correct answer.

2. Portia's high school has **3** times as many students as Lara's high school. The two high schools have a total of **2600** students. How many students does Portia's high school have?

A 600

B 650

C 1950

D 2000

E 2050

Solution:

Let x be the number of students in Lara's high school. Then Portia's high school has $3x$ students.

Therefore,

$$3x + x = 2600$$

$$x = 650.$$

Then $3x = 1950$.

Thus, **C** is the correct answer.

3. The sum of two natural numbers is 17,402. One of the two numbers is divisible by 10. If the units digit of that number is erased, the other number is obtained. What is the difference of these two numbers?

A 10,272

B 11,700

C 13,362

D 14,238

E 15,426

Solution:

Let x and y be the two numbers. WLOG, let x be divisible by 10. Then the units digit of x is 0.

If we erase the units digit, then we are essentially dividing x by 10. The problem statement also gives us that $\frac{x}{10} = y$.

Therefore,

$$\begin{aligned}\frac{x}{10} + x &= 17,402 \\ \frac{11x}{10} &= 17,402 \\ x &= 15,820.\end{aligned}$$

Then

$$x - \frac{x}{10} = 14,238.$$

Thus, **D** is the correct answer.

4. A cart rolls down a hill, traveling 5 inches the first second and accelerating so that during each successive 1-second time interval, it travels 7 inches more than during the previous 1-second interval. The cart takes 30 seconds to reach the bottom of the hill. How far, in inches, does it travel?

A 215

B 360

C 2992

D 3195

E 3242

Solution:

The distance travelled every second forms an arithmetic sequence:

$$5, 5 + 7, 5 + 2 \cdot 7, \dots$$

The standard arithmetic-sequence sum formula is

$$n \cdot \frac{a_1 + a_n}{2}.$$

We know the number of terms is 30 and the first term is 5. The last term is

$$5 + 29 \cdot 7 = 208.$$

Plugging these values into the expression yields

$$30 \cdot \frac{5 + 208}{2} = 15 \cdot 213 = 3195.$$

Thus, **D** is the correct answer.

5. The quiz scores of a class with $k > 12$ students have a mean of 8. The mean of a collection of 12 of these quiz scores is 14. What is the mean of the remaining quiz scores in terms of k ?

A $\frac{14 - 8}{k - 12}$

B $\frac{8k - 168}{k - 12}$

C $\frac{14}{12} - \frac{8}{k}$

D $\frac{14(k - 12)}{k^2}$

E $\frac{14(k - 12)}{8k}$

Solution:

The sum of the scores of everyone in the class is $8k$. The sum of the scores in the collection of 12 is $12 \cdot 14 = 168$.

This means that the sum of the scores of everyone not in the collection is $8k - 168$.

There are also $k - 12$ people not in the collection. Therefore, the average is

$$\frac{8k - 168}{k - 12}.$$

Thus, **B** is the correct answer.

6. Chantal and Jean start hiking from a trailhead toward a fire tower. Jean is wearing a heavy backpack and walks slower. Chantal starts walking at 4 miles per hour. Halfway to the tower, the trail becomes really steep, and Chantal slows down to 2 miles per hour. After reaching the tower, she immediately turns around and descends the steep part of the trail at 3 miles per hour. She meets Jean at the halfway point. What was Jean's average speed, in miles per hour, until they meet?

A $\frac{12}{13}$

B 1

C $\frac{13}{12}$

D $\frac{24}{13}$

E 2

Solution:

Let $2d$ be the distance from the trailhead to the fire tower, where $d > 0$.

Then Chantal hiked for

$$\frac{d}{4} + \frac{d}{2} + \frac{d}{3} = \frac{13d}{12}$$

hours.

If Jean travelled d miles in $\frac{13d}{12}$ hours, then his speed was

$$d \div \frac{13d}{12} = \frac{12}{13}$$

miles per hour.

Thus, **A** is the correct answer.

7. Tom has a collection of 13 snakes, 4 of which are purple and 5 of which are happy. He observes that

- all of his happy snakes can add,
- none of his purple snakes can subtract, and
- all of his snakes that can't subtract also can't add.

Which of these conclusions can be drawn about Tom's snakes?

- ☐ A Purple snakes can add.
- ☐ B Purple snakes are happy.
- ☐ C Snakes that can add are purple.
- ☒ D Happy snakes are not purple.
- ☐ E Happy snakes can't subtract.

Solution:

Note that the third condition ensures that purple snakes can't add.

We also know that all happy snakes can add, which means that happy snakes can't be purple as well.

Thus, **D** is the correct answer.

8. When a student multiplied the number 66 by the repeating decimal,

$$\underline{1}.\underline{a} \, \underline{b} \, \underline{a} \, \underline{b} \dots = \underline{1}.\overline{\underline{a} \, \underline{b}},$$

where a and b are digits, he did not notice the notation and just multiplied 66 times $\underline{1}.\underline{a} \, \underline{b}$. Later he found that his answer is 0.5 less than the correct answer. What is the 2-digit integer $\underline{a} \, \underline{b}$?

A 15

B 30

C 45

D 60

E 75

Solution:

Let $N = 10a + b$, the two-digit integer formed by the digits a and b . Then

$$1.\overline{ab} = 1 + \frac{N}{99}$$

while the terminating decimal the student used is

$$1.ab = 1 + \frac{N}{100}.$$

The correct product exceeds the student's product by 0.5, so

$$66 \left(\frac{N}{99} - \frac{N}{100} \right) = \frac{N}{150} = 0.5.$$

Hence $N = 75$.

Thus, **E** is the correct answer.

9. What is the least possible value of $(xy - 1)^2 + (x + y)^2$ for real numbers x and y ?

A 0

B $\frac{1}{4}$

C $\frac{1}{2}$

D 1

E 2

Solution:

Expanding, we get

$$\begin{aligned}x^2y^2 - 2xy + 1 + x^2 + 2xy + y^2 \\= x^2y^2 + x^2 + y^2 + 1.\end{aligned}$$

Note that every square must be non-negative. Therefore, the minimum value is when all the terms except 1 are 0, making the sum 1.

This is attainable when $x = y = 0$.

Thus, **D** is the correct answer.

10. Which of the following is equivalent to

$$\begin{aligned} &(2 + 3)(2^2 + 3^2) \\ &\cdot (2^4 + 3^4)(2^8 + 3^8) \\ &\cdot (2^{16} + 3^{16})(2^{32} + 3^{32}) \\ &\cdot (2^{64} + 3^{64})? \end{aligned}$$

A $3^{127} + 2^{127}$

B $3^{127} + 2^{127} + 2 \cdot 3^{63} + 3 \cdot 2^{63}$

C $3^{128} - 2^{128}$

D $3^{128} + 2^{128}$

E 5^{127}

Solution:

Multiply the product by $3 - 2 = 1$. Repeatedly applying the difference-of-squares identity gives

$$\begin{aligned} (3 - 2)(3 + 2) &= 3^2 - 2^2, \\ (3^2 - 2^2)(3^2 + 2^2) &= 3^4 - 2^4, \end{aligned}$$

and the same cancellation continues through the final factor. Therefore the product is $3^{128} - 2^{128}$.

Thus, **C** is the correct answer.

11. For which of the following integers b is the base- b number $2021_b - 221_b$ not divisible by 3?

- ☐ A 3
- ☐ B 4
- ☐ C 6
- ☐ D 7
- ☒ E 8

Solution:

We can express this expression in base 10 using the definition of bases:

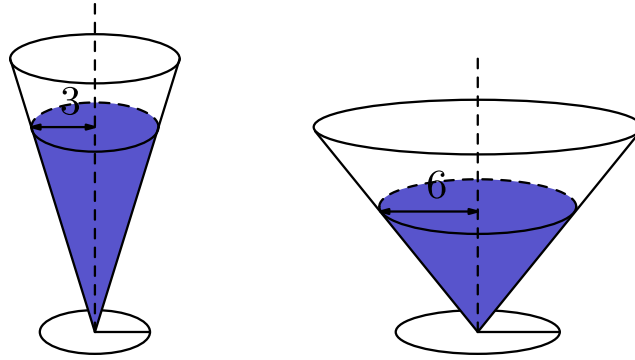
$$\begin{aligned} 2b^3 + 2b + 1 - 2b^2 - 2b - 1 \\ = 2b^3 - 2b^2 = 2b^2(b - 1). \end{aligned}$$

For this to be divisible by 3, either b or $b - 1$ must be divisible by 3.

The only answer choice that satisfies neither of these conditions is 8.

Thus, **E** is the correct answer.

12. Two right circular cones with vertices facing down as shown in the figure below contain the same amount of liquid. The radii of the tops of the liquid surfaces are 3 cm and 6 cm. Into each cone is dropped a spherical marble of radius 1 cm, which sinks to the bottom and is completely submerged without spilling any liquid. What is the ratio of the rise of the liquid level in the narrow cone to the rise of the liquid level in the wide cone?



- A 1 : 1
- B 47 : 43
- C 2 : 1
- D 40 : 13
- E 4 : 1**

Solution:

Let the initial liquid heights in the narrow and wide cones be h_1 and h_2 . Since the liquid volumes are equal,

$$\frac{1}{3}\pi(3)^2h_1 = \frac{1}{3}\pi(6)^2h_2,$$

so $h_1 = 4h_2$.

After the identical marbles are dropped in, each cone must contain the same final volume below the liquid surface: the original liquid volume plus the volume of one marble. If the new liquid-surface radii are $3x$ and $6y$, similarity gives new heights h_1x and h_2y . Thus

$$\frac{1}{3}\pi(3x)^2h_1x = \frac{1}{3}\pi(6y)^2h_2y.$$

Using $h_1 = 4h_2$, this simplifies to $x^3 = y^3$, so $x = y$. The rise ratio is therefore

$$\begin{aligned}h_1(x - 1) : h_2(y - 1) &= h_1 : h_2 \\ &= 4 : 1.\end{aligned}$$

Thus, **E** is the correct answer.

13. What is the volume of tetrahedron $ABCD$ with edge lengths $AB = 2$, $AC = 3$, $AD = 4$, $BC = \sqrt{13}$, $BD = 2\sqrt{5}$, and $CD = 5$?

A 3

B $2\sqrt{3}$

C 4

D $3\sqrt{3}$

E 6

Solution:

Place $A = (0, 0, 0)$, $B = (2, 0, 0)$, $C = (0, 3, 0)$, and $D = (0, 0, 4)$. Then

$$BC = \sqrt{2^2 + 3^2} = \sqrt{13},$$

$$BD = \sqrt{2^2 + 4^2} = 2\sqrt{5},$$

$$CD = \sqrt{3^2 + 4^2} = 5,$$

so this coordinate model matches all the given edge lengths. The tetrahedron is a rectangular-corner tetrahedron with perpendicular edge lengths 2, 3, 4 from A , so its volume is

$$\frac{1}{6}(2)(3)(4) = 4.$$

Thus, **C** is the correct answer.

14. All the roots of the polynomial

$$z^6 - 10z^5 + Az^4 + Bz^3 + Cz^2 + Dz + 16$$

are positive integers, possibly repeated. What is the value of B ?

A ☒ -88

B ☐ -80

C ☐ -64

D ☐ -41

E ☐ -40

Solution:

By Vieta's formulas, the six roots have sum 10 and product 16. Because the product is a power of 2, every positive integer root is a power of 2. Distributing the four factors of 2 among six roots gives the least possible sum when four roots are 2 and two roots are 1; that sum is already 10. Hence the roots are

$$1, 1, 2, 2, 2, 2.$$

The coefficient B is the negative of the sum of all products of three roots. Choosing zero, one, or two of the two roots equal to 1 gives

$$\begin{aligned} B &= - \left(\binom{4}{3} 2^3 + 2 \binom{4}{2} 2^2 \right. \\ &\quad \left. + \binom{4}{1} 2 \right) \\ &= -88. \end{aligned}$$

Thus, **A** is the correct answer.

15. Values for A, B, C , and D are to be selected from $\{1, 2, 3, 4, 5, 6\}$ without replacement (i.e., no two letters have the same value). How many ways are there to make such choices so that the two curves $y = Ax^2 + B$ and $y = Cx^2 + D$ intersect?

(The order in which the curves are listed does not matter; for example, the choices $A = 3, B = 2, C = 4, D = 1$ is considered the same as the choices $A = 4, B = 1, C = 3, D = 2$.)

- ☐ A 30
- ☐ B 60
- ☒ C 90
- ☐ D 180
- ☐ E 360

Solution:

Setting the equations equal to each other, we get

$$Ax^2 + B = Cx^2 + D$$

$$x^2(A - C) = D - B$$

$$x^2 = \frac{D - B}{A - C} \geq 0$$

since squares are non-negative.

This means $D - B$ and $A - C$ must both have the same sign.

If we choose two distinct values for (A, C) and (B, D) , there are 2 ways to arrange them such that the numerator and denominator both have the same sign.

We have to divide by 2, however, since the two curves are not considered distinct.

Therefore, the total number of tuples is

$$\frac{1}{2} \binom{6}{2} \binom{4}{2} \cdot 2 = 90.$$

Thus, **C** is the correct answer.

16. In the following list of numbers, the integer n appears n times in the list for $1 \leq n \leq 200$.

1, 2, 2, 3, 3, 3, 4, 4, 4, 4, ...,
200, 200, ..., 200

What is the median of the numbers in this list?

- A 100.5
- B 134
- C 142**
- D 150.5
- E 167

Solution:

The list contains

$$1 + 2 + \cdots + 200 = \frac{200 \cdot 201}{2} \\ = 20100$$

entries, so its two middle positions are 10050 and 10051.

There are

$$\frac{141 \cdot 142}{2} = 10011$$

entries through the last 141, and

$$\frac{142 \cdot 143}{2} = 10153$$

entries through the last 142. Thus both middle entries are 142, so the median is 142.

Thus, **C** is the correct answer.

17. Trapezoid $ABCD$ has $\overline{AB} \parallel \overline{CD}$, $BC = CD = 43$, and $\overline{AD} \perp \overline{BD}$. Let O be the intersection of the diagonals $\overline{AC}$ and $\overline{BD}$, and let P be the midpoint of $\overline{BD}$.

Given that $OP = 11$, the length of AD can be written in the form $m\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. What is $m + n$?

A 65

B 132

C 157

D 194

E 215

Solution:

Because $BC = CD$, the median from C to BD is perpendicular to BD . Thus $\triangle BPC$ is a right triangle. Let $\angle DBC = \alpha$. Since $AB \parallel CD$, we also have $\angle ABD = \alpha$, so $\triangle BPC \sim \triangle BDA$.

Since P is the midpoint of BD , we have $\frac{BD}{BP} = 2$. In the similarity, BC corresponds to AB , so

$$\begin{aligned}\frac{AB}{BC} &= 2, \\ AB &= 2 \cdot 43 = 86.\end{aligned}$$

Also, $\triangle ABO \sim \triangle CDO$, so

$$\frac{BO}{OD} = \frac{AB}{CD} = 2.$$

Since $OP = 11$ and P is the midpoint of BD , write $BP = PD = t$. Then $BO = t + 11$ and $OD = t - 11$, so

$$\frac{t + 11}{t - 11} = 2.$$

This gives $t = 33$, hence $BD = 66$. Finally, $\triangle ABD$ is right, so

$$\begin{aligned} AD &= \sqrt{AB^2 - BD^2} \\ &= \sqrt{86^2 - 66^2} \\ &= 4\sqrt{190}. \end{aligned}$$

Thus $m + n = 4 + 190 = 194$.

Thus, **D** is the correct answer.

18. Let f be a function defined on the set of positive rational numbers with the property that $f(a \cdot b) = f(a) + f(b)$ for all positive rational numbers a and b . Suppose that f also has the property that $f(p) = p$ for every prime number p . For which of the following numbers x is $f(x) < 0$?

A $\frac{17}{32}$

B $\frac{11}{16}$

C $\frac{7}{9}$

D $\frac{7}{6}$

E $\frac{25}{11}$

Solution:

Repeated use of the functional equation gives $f(p^e) = ef(p) = ep$ for every prime p and positive integer e . Also,

$$f(a) = f\left(\frac{a}{b}\right) + f(b),$$

$$\text{so } f\left(\frac{a}{b}\right) = f(a) - f(b).$$

Evaluating the choices by prime factorization,

$$f\left(\frac{17}{32}\right) = 17 - 5 \cdot 2 = 7,$$

$$f\left(\frac{11}{16}\right) = 11 - 4 \cdot 2 = 3,$$

$$f\left(\frac{7}{9}\right) = 7 - 2 \cdot 3 = 1,$$

$$f\left(\frac{7}{6}\right) = 7 - 2 - 3 = 2,$$

$$f\left(\frac{25}{11}\right) = 2 \cdot 5 - 11 = -1.$$

Only the final value is negative.

Thus, **E** is the correct answer.

19. The area of the region bounded by the graph of

$$x^2 + y^2 = 3|x - y| + 3|x + y|$$

is $m + n\pi$, where m and n are integers. What is $m + n$?

A 18

B 27

C 36

D 45

E 54

Solution:

Consider the four sign cases for $x - y$ and $x + y$. In one case, for example, $|x - y| = x - y$ and $|x + y| = x + y$, so

$$\begin{aligned} x^2 + y^2 &= 6x \\ \implies (x - 3)^2 + y^2 &= 9. \end{aligned}$$

The other three cases similarly give circles of radius 3 centered at $(0, 3)$, $(-3, 0)$, and $(0, -3)$. The relevant arcs form the boundary shown by these four congruent circle pieces.

The region consists of a central square of side length 6, together with four semicircles of radius 3. The square contributes area 36, and the four semicircles have the area of two full radius-3 circles, namely 18π .

Therefore the area is $36 + 18\pi$, so $m + n = 36 + 18 = 54$.

Thus, **E** is the correct answer.

20. In how many ways can the sequence 1, 2, 3, 4, 5 be rearranged so that no three consecutive terms are increasing and no three consecutive terms are decreasing?

A 10

B 18

C 24

D 32

E 44

Solution:

A permutation is valid exactly when the four comparison signs between consecutive terms alternate. Thus the signs must be either up-down-up-down or down-up-down-up.

For the up-down-up-down pattern, the largest entry 5 must be in position 2 or position 4. If it is in position 2, let the entry in position 4 be r . Its two neighbors must be distinct numbers less than r , which can be ordered in $(r - 1)(r - 2)$ ways.

Summing over $r = 1, 2, 3, 4$ gives

$$0 + 0 + 2 + 6 = 8$$

permutations. By symmetry there are another 8 when 5 is in position 4, for a total of 16 with this comparison pattern.

Replacing every entry x by $6 - x$ gives a bijection to the down-up-down-up permutations, so there are another 16.

The total number of valid rearrangements is $16 + 16 = 32$.

Thus, **D** is the correct answer.

21. Let $ABCDEF$ be an equiangular hexagon. The lines AB , CD , and EF determine a triangle with area $192\sqrt{3}$, and the lines BC , DE , and FA determine a triangle with area $324\sqrt{3}$. The perimeter of hexagon $ABCDEF$ can be expressed as $m + n\sqrt{p}$, where m , n , and p are positive integers and p is not divisible by the square of any prime. What is $m + n + p$?

A 47

B 52

C 55

D 58

E 63

Solution:

Let the intersections of lines AB , CD , EF form triangle PQR , and let the intersections of lines BC , DE , FA form triangle XYZ . Because the hexagon is equiangular, all these outer triangles are equilateral.

For an equilateral triangle with side length s , the area is $\frac{\sqrt{3}}{4}s^2$. Hence

$$\begin{aligned}\frac{\sqrt{3}}{4}PQ^2 &= 192\sqrt{3}, \\ \frac{\sqrt{3}}{4}YZ^2 &= 324\sqrt{3}.\end{aligned}$$

So $PQ = 16\sqrt{3}$ and $YZ = 36$. To justify the perimeter relation, write the consecutive hexagon side lengths as a, b, c, d, e, f . The two alternating-line triangles have side lengths $b + c + d$ and $c + d + e$, while closure of the hexagon gives $a + f = c + d$. Hence their side-length sum is

$$\begin{aligned}
& (b + c + d) \\
& + (c + d + e) \\
& = b + e + 2(c + d) \\
& = (a + b + c) + (d + e + f),
\end{aligned}$$

the hexagon's perimeter. Therefore the perimeter is

$$PQ + YZ = 16\sqrt{3} + 36.$$

Thus $m + n + p = 36 + 16 + 3 = 55$.

Thus, **C** is the correct answer.

22. Hiram's algebra notes are 50 pages long and are printed on 25 sheets of paper; the first sheet contains pages 1 and 2, the second sheet contains pages 3 and 4, and so on. One day he leaves his notes on the table before leaving for lunch, and his roommate decides to borrow some pages from the middle of the notes. When Hiram comes back, he discovers that his roommate has taken a consecutive set of sheets from the notes and that the average (mean) of the page numbers on all remaining sheets is exactly 19. How many sheets were borrowed?

A 10

B 13

C 15

D 17

E 20

Solution:

Suppose the borrowed sheets are sheets a through b , and let $s = b - a + 1$. The borrowed pages run from $2a - 1$ through $2b$, so there are $2s$ borrowed pages and their sum is $s(2a + 2b - 1)$.

The total sum of all page numbers is $\frac{50 \cdot 51}{2} = 1275$. If the remaining pages have mean 19, then

$$\begin{aligned}1275 - s(2a + 2b - 1) \\&= 19(50 - 2s), \\s(2a + 2b - 39) &= 325.\end{aligned}$$

Because $b - a + 1$ is a positive divisor of 325 and is at most 25, its only possibilities are 1, 5, 13, 25. The first two would force $b > 25$, and 25 would remove every sheet. Thus the only valid possibility is

$$\begin{aligned}2a + 2b - 39 &= 25, \\b - a + 1 &= 13.\end{aligned}$$

Thus $a + b = 32$ and $b - a = 12$, so $a = 10$ and $b = 22$. Therefore 13 sheets were borrowed.

Thus, **B** is the correct answer.

23. Frieda the frog begins a sequence of hops on a 3×3 grid of squares, moving one square on each hop and choosing at random the direction of each hop—up, down, left, or right. She does not hop diagonally. When the direction of a hop would take Frieda off the grid, she “wraps around” and jumps to the opposite edge. For example if Frieda begins in the center square and makes two hops “up”, the first hop would place her in the top row middle square, and the second hop would cause Frieda to jump to the opposite edge, landing in the bottom row middle square.

Suppose Frieda starts from the center square, makes at most four hops at random, and stops hopping if she lands on a corner square. What is the probability that she reaches a corner square on one of the four hops?

- A $\frac{9}{16}$
- B $\frac{5}{8}$
- C $\frac{3}{4}$
- D $\frac{25}{32}$**
- E $\frac{13}{16}$

Solution:

Classify a square as M for the center, E for a non-corner edge square, and C for a corner. Frieda starts at M , and the first hop always takes her to an E .

From an edge square, the probabilities of moving to C , E , M are $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{4}$, respectively. From M , the next hop always goes to an E .

Now count the possible first-hit patterns within four hops:

$$EC : 1 \cdot \frac{1}{2} = \frac{1}{2},$$

$$EEC : 1 \cdot \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8},$$

$$EEEC : \quad 1 \cdot \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{32},$$

$$EMEC : \quad 1 \cdot \frac{1}{4} \cdot 1 \cdot \frac{1}{2} = \frac{1}{8}.$$

Adding gives

$$\frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \frac{1}{8} = \frac{25}{32}.$$

Thus, **D** is the correct answer.

24. The interior of a quadrilateral is bounded by the graphs of $(x + ay)^2 = 4a^2$ and $(ax - y)^2 = a^2$, where a is a positive real number. What is the area of this region in terms of a , valid for all $a > 0$?

A $\frac{8a^2}{(a + 1)^2}$

B $\frac{4a}{a + 1}$

C $\frac{8a}{a + 1}$

D $\frac{8a^2}{a^2 + 1}$

E $\frac{8a}{a^2 + 1}$

Solution:

Note that each of the equations yields two parallel lines.

$$(x + ay)^2 = 4a^2$$

results in the two lines

$$x + ay - 2a = 0$$

and

$$x + ay + 2a = 0.$$

Both of these lines have a slope of $-\frac{1}{a}$.

Similarly,

$$(ax - y)^2 = a^2$$

results in the lines

$$ax - y - a = 0$$

and

$$ax - y + a = 0.$$

These lines have slope a .

Note that each pair of lines is perpendicular to the other pair of lines. This shows that the equations form a rectangle.

Recall that the formula for the distance d between two parallel lines

$$\begin{cases} Ax + By + C_1 = 0 \\ Ax + By + C_2 = 0 \end{cases}$$

is

$$d = \frac{|C_2 - C_1|}{\sqrt{A^2 + B^2}}.$$

Using this formula, we get that the distance between the first pair of lines is

$$\frac{4a}{\sqrt{a^2 + 1}}.$$

Similarly, the distance between the second pair of lines is

$$\frac{2a}{\sqrt{a^2 + 1}}.$$

These are the side lengths of the rectangle. Multiplying yields the area

$$\frac{8a^2}{a^2 + 1}.$$

Thus, **D** is the correct answer.

25. How many ways are there to place 3 indistinguishable red chips, 3 indistinguishable blue chips, and 3 indistinguishable green chips in the squares of a 3×3 grid so that no two chips of the same color are directly adjacent to each other, either vertically or horizontally?

A 12

B 18

C 24

D 30

E 36

Solution:

Choose the center color in 3 ways. Its other two chips cannot occupy any edge-middle square, because those squares are adjacent to the center. Thus they must occupy two of the four corners, which can be chosen in $\binom{4}{2} = 6$ ways.

For either possible corner pattern—two opposite corners or two corners on the same side—the adjacency conditions force the remaining two colors up to interchanging them. Hence there are 2 completions for each choice of the center color and its two corners. The total is

$$3 \binom{4}{2} \cdot 2 = 36.$$

Thus, **E** is the correct answer.

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