2021 AMC 10A Fall Solutions

Typeset by: LIVE by Po-Shen Loh
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1. What is the value of

$$\frac{(2112-2021)^2}{169}?$$

A 7

B 21

C 49

D 64

E 91

Solution:

Since $2112-2021=91=7\cdot 13$,

$$\begin{aligned}\frac{(2112-2021)^2}{169} &= \frac{(7\cdot 13)^2}{13^2}\\ &= 7^2 = 49.\end{aligned}$$

Thus, **C** is the correct answer.

2. Menkara has a 4×6 index card. If she shortens the length of one side of this card by 1 inch, the card would have area 18 square inches. What would the area of the card be in square inches if instead she shortens the length of the other side by 1 inch?

A 16

B 17

C 18

D 19

E 20

Solution:

If she shortens the 4-inch side by 1, she has a 3×6 card, whose area is 18 square inches.

Therefore the given shortening was on the 4-inch side. Shortening the other side instead gives a 4×5 card, with area 20 square inches.

Thus, **E** is the correct answer.

3. What is the maximum number of balls of clay with radius 2 that can completely fit inside a cube of side length 6 assuming that the balls can be reshaped but not compressed before they are packed in the cube?

A 3

B 4

C 5

D 6

E 7

Solution:

The cube has volume $6^3 = 216$. One ball of clay has volume $\frac{4}{3}\pi \cdot 2^3 = \frac{32\pi}{3}$.

Because the clay may be reshaped but not compressed, the maximum number of balls is

$$\left\lfloor \frac{216}{\frac{32\pi}{3}} \right\rfloor = \left\lfloor \frac{81}{4\pi} \right\rfloor.$$

Since $12 < 4\pi < 13$, we have $6 < \frac{81}{4\pi} < \frac{81}{12} < 7$. Therefore the floor is 6.

Thus, **D** is the correct answer.

4. Mr. Lopez has a choice of two routes to get to work. Route A is 6 miles long, and his average speed along this route is 30 miles per hour. Route B is 5 miles long, and his average speed along this route is 40 miles per hour, except for a $\frac{1}{2}$-mile stretch in a school zone where his average speed is 20 miles per hour. By how many minutes is Route B quicker than Route A?

A $2\frac{3}{4}$

B $3\frac{3}{4}$

C $4\frac{1}{2}$

D $5\frac{1}{2}$

E $6\frac{3}{4}$

Solution:

Mr. Lopez would take

$$\frac{6}{30} \cdot 60 = 12$$

minutes to travel on Route A.

On Route B, he would take

$$\left(\frac{5-0.5}{40} + \frac{0.5}{20}\right) \cdot 60 = 8.25$$

minutes.

The difference in times along these routes is $12 - 8.25 = 3.75$ minutes.

Thus, **B** is the correct answer.

5. The six-digit number $\underline{2}\,\underline{0}\,\underline{2}\,\underline{1}\,\underline{0}\,\underline{A}$ is prime for only one digit A. What is A?

A 1

B 3

C 5

D 7

E 9

Solution:

Note that A cannot be even, as then the number would be divisible by 2.

A also cannot be 5, as that would make the number divisible by 5.

If A equaled 1 or 7, then the sum of the digits of the number would be 6 and 12 respectively.

This would make the number divisible by 3, so that rules out A equaling either of these numbers.

Finally, if A equals 3, then the whole number becomes 202103. If we look at the difference of the sums of alternating digits, we get

$$2 + 2 - 1 - 3 = 0,$$

which means the number is divisible by 11.

Every choice except 9 makes the number composite. Because the problem states that exactly one digit works, that digit must be $A = 9$. (Indeed, trial division by the primes at most $\sqrt{202109} < 450$ confirms that 202109 is prime.)

Thus, **E** is the correct answer.

6. Elmer the emu takes 44 equal strides to walk between consecutive telephone poles on a rural road. Oscar the ostrich can cover the same distance in 12 equal leaps. The telephone poles are evenly spaced, and the 41st pole along this road is exactly one mile (5280 feet) from the first pole. How much longer, in feet, is Oscar's leap than Elmer's stride?

A 6

B 8

C 10

D 11

E 15

Solution:

There are 40 gaps between the 1st and 41st pole, which means that the distance between consecutive poles is

$$5280 \div 40 = 132$$

feet.

This means that each of Elmer's strides is

$$132 \div 44 = 3$$

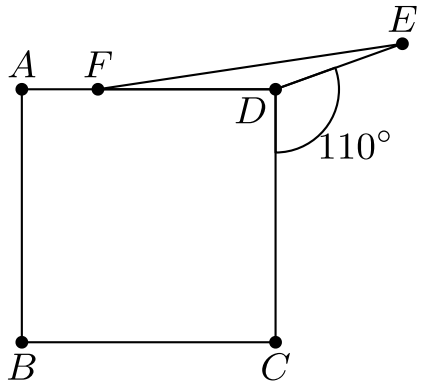
feet. Similarly, each of Oscar's leaps is

$$132 \div 12 = 11$$

feet. This makes Oscar's leap $11 - 3 = 8$ feet longer.

Thus, **B** is the correct answer.

7. As shown in the figure below, point E lies in the opposite half-plane determined by line CD from point A so that $\angle CDE = 110^\circ$. Point F lies on $\overline{AD}$ so that $DE = DF$, and $ABCD$ is a square. What is the degree measure of $\angle AFE$?



A 160

B 164

C 166

D 170

E 174

Solution:

Since $\angle ADC = 90^\circ$, we get that

$$\angle FDE = 360^\circ - 90^\circ - 110^\circ$$

$$= 160^\circ.$$

Also since $\triangle FDE$ is isosceles, we get that

$$\angle EFD = \frac{180^\circ - 160^\circ}{2} = 10^\circ.$$

Finally, we get that

$$\angle AFE = 180^\circ - 10^\circ = 170^\circ.$$

Thus, **D** is the correct answer.

8. A two-digit positive integer is said to be *cuddly* if it is equal to the sum of its nonzero tens digit and the square of its units digit. How many two-digit positive integers are cuddly?

A 0

B 1

C 2

D 3

E 4

Solution:

Let $\underline{a}\ \underline{b}$ be a 2-digit cuddly number.

Then

$$10a + b = a + b^2.$$

Rearranging, we get

$$9a = b(b-1).$$

Because consecutive integers cannot both be divisible by 3, one of b and $b-1$ must contain both factors of 3.

For a digit b, this leaves $b = 0, 1,$ or 9. The first two choices give $a = 0$, which does not make a two-digit number. If $b = 9$, then $a = 8$. Checking, we get that 89 is a *cuddly* number. This shows that there is only 1 two-digit *cuddly* number.

Thus, **B** is the correct answer.

9. When a certain unfair die is rolled, an even number is 3 times as likely to appear as an odd number. The die is rolled twice. What is the probability that the sum of the numbers rolled is even?

A $\frac{3}{8}$

B $\frac{4}{9}$

C $\frac{5}{9}$

D $\frac{9}{16}$

E $\frac{5}{8}$

Solution:

Let p be the probability that an odd number is rolled. Then $3p$ is the probability an even number is rolled. We know that

$$p + 3p = 1 \Rightarrow p = \frac{1}{4}.$$

The only way for the sum to be even is if both rolls have the same parity. This happens with a probability of

$$\frac{1}{4}^2 + \frac{3}{4}^2 = \frac{10}{16} = \frac{5}{8}.$$

Thus, **E** is the correct answer.

10. A school has 100 students and 5 teachers. In the first period, each student is taking one class, and each teacher is teaching one class. The enrollments in the classes are $50, 20, 20, 5,$ and 5. Let t be the average value obtained if a teacher is picked at random and the number of students in their class is noted. Let s be the average value obtained if a student was picked at random and the number of students in their class, including the student, is noted. What is $t - s$?

A -18.5

B -13.5

C 0

D 13.5

E 18.5

Solution:

Recall the expected-value formula

$$\mathbb{E}[X] = \sum_x x \Pr(X = x).$$

Therefore,

$$\begin{aligned} t &= \frac{1}{5}(50 + 20 + 20 + 5 + 5) \\ &= \frac{1}{5} \cdot 100 = 20 \end{aligned}$$

and

$$\begin{aligned} s &= 50 \cdot \frac{50}{100} + 20 \cdot \frac{20}{100} \\ &+20 \cdot \frac{20}{100} + 5 \cdot \frac{5}{100} + 5 \cdot \frac{5}{100} \\ &= 25 + 4 + 4 + 0.25 + 0.25 \\ &= 33.5 \end{aligned}$$

$t - s = -13.5.$

Thus, **B** is the correct answer.

11. Emily sees a ship traveling at a constant speed along a straight section of a river. She walks parallel to the riverbank at a uniform rate faster than the ship. She counts 210 equal steps walking from the back of the ship to the front. Walking in the opposite direction, she counts 42 steps of the same size from the front of the ship to the back. In terms of Emily's equal steps, what is the length of the ship?

A 70

B 84

C 98

D 105

E 126

Solution:

Let x be the length of the ship. Then in the time that Emily moves 210 steps, the ship moves $210 - x$ steps.

In the time that Emily moves 42 steps, the ship moves $x - 42$ steps. Since the ship and Emily move at a constant rate

$$\frac{210}{210 - x} = \frac{42}{x - 42}.$$

Cross-multiplying yields

$$210x - 210 \cdot 42 = 210 \cdot 42 - 42x$$

$$42 \cdot 6x = 2 \cdot 42 \cdot 210$$

$$x = 70.$$

Thus, **A** is the correct answer.

12. The base-nine representation of the number N is $27{,}006{,}000{,}052_9$. What is the remainder when N is divided by 5?

A 0

B 1

C 2

D 3

E 4

Solution:

Note that

$$9 \equiv -1 \pmod 5.$$

Then if expand N using the definition of bases, we get

$$\begin{aligned} N &= 2 \cdot 9^{10} + 7 \cdot 9^9 + 6 \cdot 9^6 + \\ &\quad 5 \cdot 9 + 2 \\ &\equiv 2(-1)^{10} + 7(-1)^9 + 6(-1)^6 + \\ &\quad 5(-1) + 2 \pmod 5 \\ &\equiv 2 - 7 + 6 - 5 + 2 \pmod 5 \\ &\equiv 3 \pmod 5. \end{aligned}$$

Thus, **D** is the correct answer.

13. Each of 6 balls is randomly and independently painted either black or white with equal probability. What is the probability that every ball is different in color from more than half of the other 5 balls?

A $\frac{1}{64}$

B $\frac{1}{6}$

C $\frac{1}{4}$

D $\frac{5}{16}$

E $\frac{1}{2}$

Solution:

Note that for this restriction to hold, there must be 3 balls of each color.

There are $2^6 = 64$ ways to color the balls and $\binom{6}{3} = 20$ to choose which balls are white.

The desired probability is therefore $\frac{20}{64} = \frac{5}{16}$.

Thus, **D** is the correct answer.

14. How many ordered pairs (x, y) of real numbers satisfy the following system of equations?

$$x^2 + 3y = 9,$$
$$(|x| + |y| - 4)^2 = 1.$$

A 1

B 2

C 3

D 5

E 7

Solution:

Put $t = |x|$. The second equation gives $t + |y| = 3$ or 5, while the first gives $y = 3 - \frac{t^2}{3}$.

If $y \geq 0$, then $0 \leq t \leq 3$. For $t + y = 3$, substitution gives $t - \frac{t^2}{3} = 0$, so $t = 0$ or 3. These yield $x = 0, 3, -3$, for three points. For $t + y = 5$, substitution gives $t^2 - 3t + 6 = 0$, which has no real root.

If $y < 0$, then $t > 3$. The equation $t - y = 3$ gives $t^2 + 3t - 18 = 0$, whose only nonnegative root is the excluded boundary value $t = 3$. The equation $t - y = 5$ gives $t^2 + 3t - 24 = 0$, with exactly one positive root $t = \frac{-3+\sqrt{105}}{2} > 3$. It yields two points, one for each sign of x.

The total number of ordered pairs is $3 + 2 = 5$.

Thus, **D** is the correct answer.

15. Isosceles triangle ABC has $AB = AC = 3\sqrt{6}$, and a circle with radius $5\sqrt{2}$ is tangent to line AB at B and to line AC at C. What is the area of the circle that passes through vertices A, B, and C?

A 24π

B 25π

C 26π

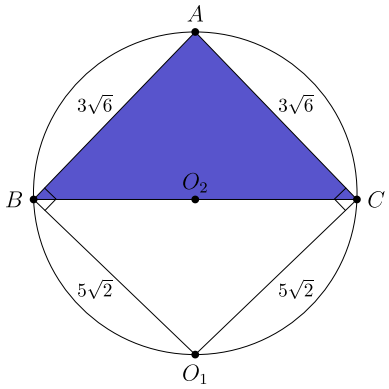
D 27π

E 28π

Solution:

Let O_1 be the center of the circle tangent to AB and AC. Then $\angle ABO_1 = \angle ACO_1 = 90^\circ$, so A, B, O_1, C are concyclic.

Because the right angles at B and C subtend AO_1, the segment AO_1 is a diameter of this circle. Let O_2 be its center. The same circle passes through A, B, and C, so it is the desired circumcircle.



By the Pythagorean Theorem in $\triangle ABO_1$,

$$\begin{aligned} AO_1 &= \sqrt{AB^2 + BO_1^2} \\ &= \sqrt{54 + 50} = 2\sqrt{26}. \end{aligned}$$

Therefore, the circumradius is $\sqrt{26}$, and the requested area is 26π.

Thus, **C** is the correct answer.

16. The graph of

$$f(x) = ||\lfloor x \rfloor| - |\lfloor 1 - x \rfloor||$$

is symmetric about which of the following? (Here $\lfloor x \rfloor$ is the greatest integer not exceeding x.)

A the y-axis

B the line $x = 1$

C the origin

D the point $\left(\frac{1}{2}, 0\right)$

E the point $(1, 0)$

Solution:

For every real x,

$$f(1 - x) = ||\lfloor 1 - x \rfloor| - |\lfloor x \rfloor|| = -f(x).$$

Equivalently, replacing x by $\frac{1}{2} + t$ gives $f\left(\frac{1}{2} - t\right) = -f\left(\frac{1}{2} + t\right)$. This is point symmetry about $\left(\frac{1}{2}, 0\right)$.

Thus, **D** is the correct answer.

17. An architect is building a structure that will place vertical pillars at the vertices of regular hexagon $ABCDEF$, which is lying horizontally on the ground. The six pillars will hold up a flat solar panel that will not be parallel to the ground. The heights of pillars at A, B, and C are 12, 9, and 10 meters, respectively. What is the height, in meters, of the pillar at E?

A 9

B $6\sqrt{3}$

C $8\sqrt{3}$

D 17

E $12\sqrt{3}$

Solution:

Put a regular hexagon in coordinates with $A = (-1, 0), B = (-\frac{1}{2}, \frac{\sqrt{3}}{2}), C = (\frac{1}{2}, \frac{\sqrt{3}}{2})$, and $E = (\frac{1}{2}, -\frac{\sqrt{3}}{2})$. Because the solar panel is flat, the height is an affine function $h(x, y) = ux + vy + w$.

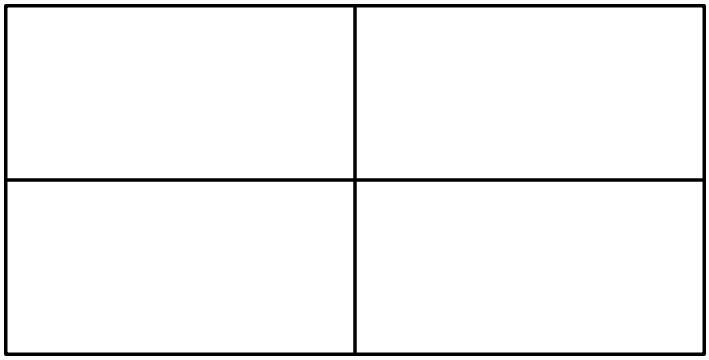
From $h(A) = 12, h(B) = 9$, and $h(C) = 10$, subtracting the last two equations gives $u = 1$. Then $-u + w = 12$, so $w = 13$. Using $h(B) = 9$ gives $-\frac{1}{2} + \frac{\sqrt{3}}{2}v + 13 = 9$, so $\sqrt{3}v = -7$.

Therefore

$$\begin{aligned} h(E) &= \frac{1}{2} - \frac{\sqrt{3}}{2}v + 13 \\ &= \frac{1}{2} + \frac{7}{2} + 13 \\ &= 17. \end{aligned}$$

Thus, **D** is the correct answer.

18. A farmer's rectangular field is partitioned into 2 by 2 grid of 4 rectangular sections as shown in the figure. In each section the farmer will plant one crop: corn, wheat, soybeans, or potatoes. The farmer does not want to grow corn and wheat in any two sections that share a border, and the farmer does not want to grow soybeans and potatoes in any two sections that share a border. Given these restrictions, in how many ways can the farmer choose crops to plant in each of the four sections of the field?



A 12

B 64

C 84

D 90

E 144

Solution:

There are 2 cases.

Case 1 : the top-right and bottom-left sections have the same crop

There are 4 options for which crop is in those sections. The other two sections have 3 options for the crop since each crop has one restriction for crops next to it. This gives us

$$4 \cdot 3 \cdot 3 = 36$$

combinations.

Case 2 : the top-right and bottom-left sections have different crops

There are 4 options for the top-right crop and 3 different options for the bottom-left crop. Each of the other two sections borders both of these crops, leaving 2 choices for each. This gives us another

$$4 \cdot 3 \cdot 2 \cdot 2 = 48$$

configurations.

In total, there are $36 + 48 = 84$ combinations.

Thus, **C** is the correct answer.

19. A disk of radius 1 rolls all the way around the inside of a square of side length $s > 4$ and sweeps out a region of area A. A second disk of radius 1 rolls all the way around the outside of the same square and sweeps out a region of area $2A$. The value of s can be written as $a + \dfrac{b\pi}{c}$, where a, b, and c are positive integers and b and c are relatively prime. What is $a + b + c$?

A 10

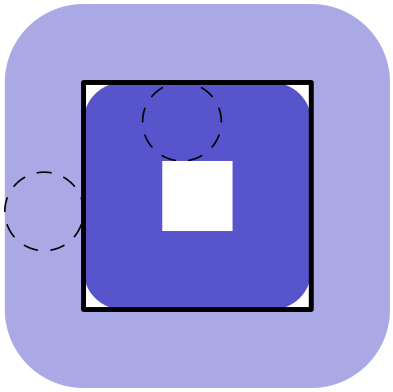
B 11

C 12

D 13

E 14

Solution:



The side length of the inner square traced out by the inner circle is $s - 4$.

There are also the small pieces remaining in the corner. These form a total area of

$$(1+1)^2 - \pi 1^2 = 4 - \pi.$$

Therefore,

$$A = s^2 - (s-4)^2 - (4-\pi)$$

$$= 8s - 20 + \pi.$$

The outer disk traces out an area that is comprised of 4 rectangles and 4 quarter-circles. The rectangles have area $s \cdot 2 = 2s$ and the quarter-circles form a circle with radius 2 and area 4π.

This gives us

$$2A = 8s + 4\pi.$$

Equating the two equations we get

$$8s + 4\pi = 2(8s - 20 + \pi).$$

Solving yields

$$8s = 40 + 2\pi$$

$$s = 5 + \frac{\pi}{4}.$$

Thus, **A** is the correct answer.

20. For how many ordered pairs (b, c) of positive integers does neither $x^2 + bx + c = 0$ nor $x^2 + cx + b = 0$ have two distinct real solutions?

A 4

B 6

C 8

D 10

E 12

Solution:

A quadratic fails to have two distinct real solutions exactly when its discriminant is nonpositive. Thus we need

$$b^2 - 4c \leq 0 \qquad c^2 - 4b \leq 0,$$

or $b^2 \leq 4c$ and $c^2 \leq 4b$.

From $b^2 \leq 4c$, we get $b^4 \leq 16c^2$. Combining this with $c^2 \leq 4b$ gives $b^4 \leq 64b$, so $b \leq 4$.

Now check $b = 1, 2, 3, 4$. The inequalities give respectively

$$(c = 1, 2), \quad (c = 1, 2),$$
$$(c = 3), \quad (c = 4).$$

There are $2 + 2 + 1 + 1 = 6$ ordered pairs.

Thus, **B** is the correct answer.

21. Each of 20 balls is tossed independently and at random into one of the 5 bins. Let p be the probability that some bin ends up with 3 balls, another with 5 balls, and the other three with 4 balls each. Let q be the probability that every bin ends up with 4 balls. What is $\frac{p}{q}$?

A 1

B 4

C 8

D 12

E 16

Solution:

All 5^{20} assignments of the distinguishable balls to the labeled bins are equally likely. For q, the number of assignments is

$$\frac{20!}{(4!)^5}.$$

For p, choose the bin with 3 balls and the bin with 5 balls in $5 \cdot 4$ ways. The number of assignments is then

$$5 \cdot 4 \cdot \frac{20!}{3!5!(4!)^3}.$$

The common probability denominator cancels, so

$$\frac{p}{q} = 20 \cdot \frac{(4!)^2}{3!5!} = 20 \cdot \frac{4}{5} = 16.$$

Thus, **E** is the correct answer.

22. Inside a right circular cone with base radius 5 and height 12 are three congruent spheres with radius r. Each sphere is tangent to the other two spheres and also tangent to the base and side of the cone. What is r?

A $\frac{3}{2}$

B $\frac{90-40\sqrt{3}}{11}$

C 2

D $\frac{144-25\sqrt{3}}{44}$

E $\frac{5}{2}$

Solution:

Let the cone have base in the plane $z=0$, center at the origin, and vertex on the z-axis. The centers of the three spheres form an equilateral triangle of side $2r$, so one sphere center may be taken at horizontal distance $\frac{2r}{\sqrt{3}}$ from the cone axis and height r above the base.

In the axial cross-section through that center and the cone axis, the side of the cone is the line $12\rho+5z=60$, where ρ is horizontal distance from the axis. The distance from $\left(\frac{2r}{\sqrt{3}},r\right)$ to this line must be r:

$$r=\frac{60-12\cdot\frac{2r}{\sqrt{3}}-5r}{13}.$$

Thus $(18+8\sqrt{3})r=60$, so

$$r=\frac{60}{18+8\sqrt{3}}=\frac{90-40\sqrt{3}}{11}.$$

Thus, **B** is the correct answer.

23. For each positive integer n, let $f_1(n)$ be twice the number of positive integer divisors of n, and for $j \geq 2$, let $f_j(n) = f_1(f_{j-1}(n))$. For how many values of $n \leq 50$ is $f_{50}(n) = 12$?

A 7

B 8

C 9

D 10

E 11

Solution:

The value 12 is fixed by the function, since 12 has 6 positive divisors and therefore $f_1(12) = 12$.

First find all $n \leq 50$ with $f_1(n) = 12$, meaning n has 6 divisors. These are

$$12, 18, 20, 28, 32, 44, 45, 50.$$

Now check whether $f_1(n)$ can be one of these values before reaching 12. Since $f_1(n)$ is twice a divisor count, the only useful possibilities in that list are 18 and 20, meaning n has 9 or 10 divisors.

For $n \leq 50$, the additional possibilities are 36, which has 9 divisors, and 48, which has 10 divisors. Therefore there are $8 + 2 = 10$ values of n.

Thus, **D** is the correct answer.

24. Each of the 12 edges of a cube is labeled 0 or 1. Two labelings are considered different even if one can be obtained from the other by a sequence of one or more rotations and/or reflections. For how many such labelings is the sum of the labels on the edges of each of the 6 faces of the cube equal to 2?

A 8

B 10

C 12

D 16

E 20

Solution:

Label one face $ABCD$. Each face must contain two 0s and two 1s, so first split by the pattern on the four edges of $ABCD$.

Case 1 : opposite edges of $ABCD$ have the same label. There are 2 such patterns on $ABCD$. For either pattern, once the label of one vertical edge, say AE, is chosen, the face-sum conditions force all remaining labels. This gives $2 \cdot 2 = 4$ labelings.

Case 2 : opposite edges of $ABCD$ have different labels. There are 4 such patterns on $ABCD$. For each, the labels of two adjacent vertical edges may be chosen in 4 ways, and then the remaining labels are forced by the face-sum conditions. This gives $4 \cdot 4 = 16$ labelings.

The total is $4 + 16 = 20$.

Thus, **E** is the correct answer.

25. A quadratic polynomial with real coefficients and leading coefficient 1 is called *disrespectful* if the equation $p(p(x)) = 0$ is satisfied by exactly three real numbers. Among all the disrespectful quadratic polynomials, there is a unique such polynomial $\tilde{p}(x)$ for which the sum of the roots is maximized. What is $\tilde{p}(1)$?

A $\frac{5}{16}$

B $\frac{1}{2}$

C $\frac{5}{8}$

D 1

E $\frac{9}{8}$

Solution:

The polynomial must have two distinct real roots: a repeated real root produces at most two real solutions of $p(p(x)) = 0$, while nonreal roots produce none. Let its roots be r and s, so

$$\begin{aligned} p(x) &= (x - r)(x - s) \\ &= x^2 - (r + s)x + rs. \end{aligned}$$

The equation $p(p(x)) = 0$ is equivalent to $p(x) = r$ or $p(x) = s$.

For exactly three real solutions, one of these two quadratic equations must have a double root and the other must have two distinct real roots. Suppose $p(x) = r$ has the double root. Its discriminant is

$$\begin{aligned} &(r + s)^2 - 4(rs - r) \\ &= (r - s)^2 + 4r, \end{aligned}$$

so $(r - s)^2 = -4r$, forcing $r \leq 0$.

The other equation, $p(x) = s$, has discriminant $(r - s)^2 + 4s = -4r + 4s = 4(s - r)$, which must be positive. Hence $s > r$, so $r - s = -2\sqrt{-r}$ and $s = r + 2\sqrt{-r}$.

The sum of the roots is $r + s = 2r + 2\sqrt{-r}$. Let $u = \sqrt{-r}$, so this is $-2u^2 + 2u$, maximized at $u = \frac{1}{2}$. Thus $r = -\frac{1}{4}$ and $s = \frac{3}{4}$.

Therefore $p(x) = x^2 - \frac{1}{2}x - \frac{3}{16}$, and

$$p(1) = 1 - \frac{1}{2} - \frac{3}{16} = \frac{5}{16}.$$

Thus, **A** is the correct answer.

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