

2022 AMC 10A Solutions

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1. What is the value of

$$3 + \frac{1}{3 + \frac{1}{3 + \frac{1}{3}}}$$

A $\frac{31}{10}$

B $\frac{49}{15}$

C $\frac{33}{10}$

D $\frac{109}{33}$

E $\frac{15}{4}$

Solution:

We can simplify this expression as follows:

$$\begin{aligned}
& 3 + \frac{1}{3 + \frac{1}{3 + \frac{1}{3}}} \\
&= 3 + \frac{1}{3 + \frac{1}{\frac{10}{3}}} \\
&= 3 + \frac{1}{3 + \frac{3}{10}} \\
&= 3 + \frac{1}{\frac{33}{10}} \\
&= 3 + \frac{10}{33} \\
&= \frac{109}{33}.
\end{aligned}$$

Thus, **D** is the correct answer.

2. Mike cycled 15 laps in 57 minutes. Assume he cycled at a constant speed throughout. Approximately how many laps did he complete in the first 27 minutes?

A 5

B 7

C 9

D 11

E 13

Solution:

We can set up a proportion to solve this problem:

$$\frac{15}{57} = \frac{x}{27}.$$

Cross multiplying, we get

$$x = \frac{15}{57} \cdot 27 = \frac{135}{19} \approx 7.$$

Thus, **B** is the correct answer.

3. The sum of three numbers is 96. The first number is 6 times the third number, and the third number is 40 less than the second number. What is the absolute value of the difference between the first and second numbers?

A 1

B 2

C 3

D 4

E 5

Solution:

Let x , y , and z be the three numbers. The conditions from the problem give us the following relations:

$$x + y + z = 96 \quad (1)$$

$$x = 6z \quad (2)$$

$$z = y - 40 \quad (3).$$

Rearranging (3), we get $y = z + 40$. Plugging this new equation and (2) into (1), we get

$$6z + z + 40 + z = 96$$

$$8z + 40 = 96$$

$$8z = 56 \Rightarrow z = 7.$$

From this, we get that

$$x = 6 \cdot z = 6 \cdot 7 = 42$$

and

$$y = z + 40 = 7 + 40 = 47.$$

Therefore, $y - x = 47 - 42 = 5$.

Thus, **E** is the correct answer.

4. In some countries, automobile fuel efficiency is measured in liters per 100 kilometers while other countries use miles per gallon. Suppose that 1 kilometer equals m miles, and 1 gallon equals l liters. Which of the following gives the fuel efficiency in liters per 100 kilometers for a car that gets x miles per gallon?

A $\frac{x}{100lm}$

B $\frac{xlm}{100}$

C $\frac{lm}{100x}$

D $\frac{100}{xlm}$

E $\frac{100lm}{x}$

Solution:

A car that gets x miles per gallon uses $\frac{1}{x}$ gallons per mile.

Since 1 mile is $\frac{1}{m}$ kilometers, this is $\frac{m}{x}$ gallons per kilometer.

Multiplying by l liters per gallon gives $\frac{lm}{x}$ liters per kilometer, so for 100 kilometers the fuel used is $\frac{100lm}{x}$ liters.

Thus, **E** is the correct answer.

5. Square $ABCD$ has side length 1. Points P , Q , R , and S each lie on a side of $ABCD$ such that $APQCRS$ is an equilateral convex hexagon with side length s . What is s ?

A $\frac{\sqrt{2}}{3}$

B $\frac{1}{2}$

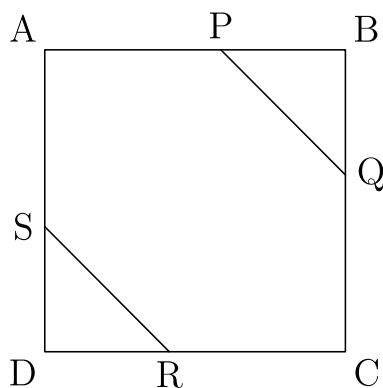
C $2 - \sqrt{2}$

D $1 - \frac{\sqrt{2}}{4}$

E $\frac{2}{3}$

Solution:

Consider the diagram:



Since $AP = QC = s$, we know that $PB = BQ$. This shows that $\triangle PBQ$ is an isosceles right triangle with hypotenuse $PQ = s$. Using the Pythagorean theorem, we get that $PB = \frac{s}{\sqrt{2}}$.

We also know that

$$1 = AB = AP + PB = s + \frac{s}{\sqrt{2}}.$$

This equation simplifies to

$$1 = \left(1 + \frac{1}{\sqrt{2}}\right)s$$

Which implies that

$$s = \frac{1}{1 + \frac{1}{\sqrt{2}}} = \frac{\sqrt{2}}{\sqrt{2} + 1}.$$

We can rationalize this fraction to get

$$\frac{\sqrt{2}}{\sqrt{2} + 1} \cdot \frac{\sqrt{2} - 1}{\sqrt{2} - 1} = 2 - \sqrt{2}.$$

Thus, **C** is the correct answer.

6. Which expression is equal to

$$\left| a - 2 - \sqrt{(a - 1)^2} \right|$$

for $a < 0$?

A $3 - 2a$

B $1 - a$

C 1

D $a + 1$

E 3

Solution:

By the definition of square root, we get that

$$\sqrt{(a - 1)^2} = |a - 1|.$$

Since $a < 0$, we get that $a - 1 < 0$, which means that $|a - 1| = 1 - a$.

The whole expression therefore simplifies to

$$|a - 2 - (1 - a)| = |2a - 3|.$$

Since $a < 0$, we know that $2a - 3 < 0$. This means that $|2a - 3| = 3 - 2a$.

Thus, **A** is the correct answer.

7. The least common multiple of a positive integer n and 18 is 180, and the greatest common divisor of n and 45 is 15. What is the sum of the digits of n ?

A 3

B 6

C 8

D 9

E 12

Solution:

Prime factorize $18 = 2 \cdot 3^2$, $45 = 3^2 \cdot 5$, and $180 = 2^2 \cdot 3^2 \cdot 5$.

The lcm condition forces the exponent of 2 in n to be 2, the exponent of 5 to be 1, the exponent of 3 to be at most 2, and every other prime exponent to be 0. The gcd condition $\gcd(n, 45) = 15 = 3 \cdot 5$ further forces the exponent of 3 in n to be exactly 1.

Therefore $n = 2^2 \cdot 3 \cdot 5 = 60$, and the sum of its digits is 6.

Thus, **B** is the correct answer.

8. A data set consists of 6 (not distinct) positive integers: 1, 7, 5, 2, 5, and X . The average (arithmetic mean) of the 6 numbers equals a value in the data set. What is the sum of all positive values of X ?

A 10

B 26

C 32

D 36

E 40

Solution:

The average of the 6 numbers is

$$\frac{1 + 7 + \dots + X}{6} = \frac{20 + X}{6}.$$

This value can equal any of the terms in the set, so we can case on what it equals.

$$\frac{20 + X}{6} = 1 \iff X = -14$$

$$\frac{20 + X}{6} = 7 \iff X = 22$$

$$\frac{20 + X}{6} = 5 \iff X = 10$$

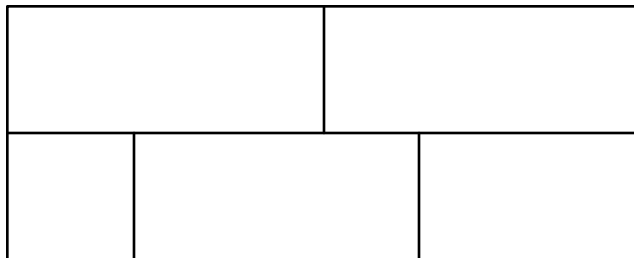
$$\frac{20 + X}{6} = 2 \iff X = -8$$

$$\frac{20 + X}{6} = X \iff X = 4$$

Adding up all the positive values for X , we get 36.

Thus, **D** is the correct answer.

9. A rectangle is partitioned into 5 regions as shown. Each region is to be painted a solid color—red, orange, yellow, blue, or green—so that regions that touch are painted different colors, and colors can be used more than once. How many different colorings are possible?



- A 120
- B 270
- C 360
- D 540**
- E 720

Solution:

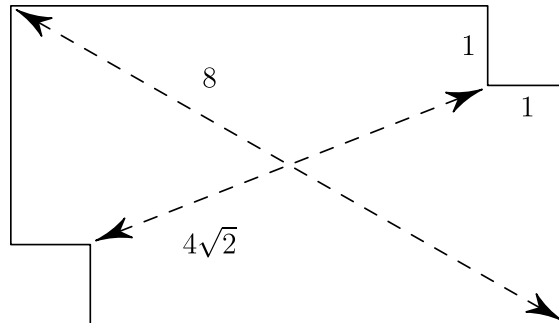
There are 5 choices for the color of the bottom left rectangle. This forces there to be 4 choices for the top left rectangle. The middle bottom rectangle touches both of the previous ones, so there are 3 color options for this rectangle.

The rectangle in the top right is also limited to 3 colors since it touches the two previous rectangles. Finally, the rectangle in the bottom right also has 3 color options.

Multiplying these together, we get $5 \cdot 4 \cdot 3 \cdot 3 \cdot 3 = 540$ total colorings.

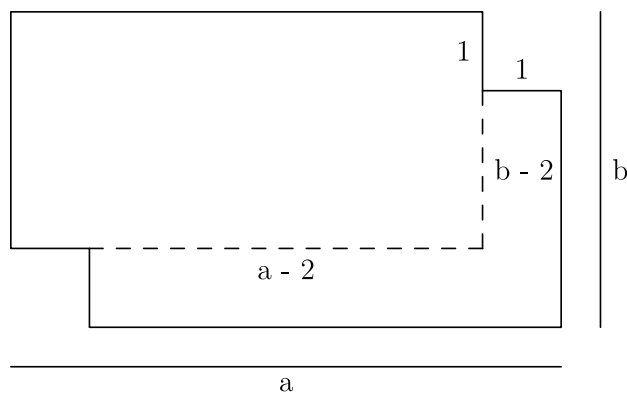
Thus, **D** is the correct answer.

10. Daniel finds a rectangular index card and measures its diagonal to be 8 centimeters. Daniel then cuts out equal squares of side 1 cm at two opposite corners of the index card and measures the distance between the two closest vertices of these squares to be $4\sqrt{2}$ centimeters, as shown below. What is the area of the original index card?



- A 14
- B $10\sqrt{2}$
- C 16
- D $12\sqrt{2}$
- E 18**

Solution:



We can label a and b as the width and height as in the diagram. Then we get that

$$a^2 + b^2 = 64$$

and

$$(a - 2)^2 + (b - 2)^2 = 32.$$

The latter expression simplifies to

$$a^2 + b^2 - 4a - 4b + 4 + 4 = 32,$$

which is the same as

$$72 - 4(a + b) = 32.$$

From this we get

$$a + b = 10.$$

Squaring this, we get

$$a^2 + b^2 + 2ab = 100,$$

which gets us that

$$2ab = 36,$$

which means that the area (ab) is 18.

Thus, **E** is the correct answer.

11. Ted mistakenly wrote

$$2^m \cdot \sqrt{\frac{1}{4096}}$$

as

$$2 \cdot \sqrt[m]{\frac{1}{4096}}.$$

What is the sum of all real numbers m for which these two expressions have the same value?

- ☐ A 5
- ☐ B 6
- ☒ C 7
- ☐ D 8
- ☐ E 9

Solution:

We can rewrite 4096 as 2^{12} , so $\frac{1}{4096} = 2^{-12}$. Then if we equate the given expressions, we get

$$2^m \cdot 2^{-6} = 2 \cdot 2^{\frac{-12}{m}}.$$

Equating the exponents, we get

$$m - 6 = 1 + \frac{-12}{m}.$$

Multiplying by m , we get

$$m^2 - 6m = m - 12$$

and so

$$m^2 - 7m + 12 = 0$$

$$(m - 4)(m - 3)$$

$$m = 4, m = 3$$

Therefore, we can see that the sum of the solutions is 7.

Thus, **C** is the correct answer.

12. On Halloween 31 children walked into the principal's office asking for candy. They can be classified into three types: Some always lie; some always tell the truth; and some alternately lie and tell the truth. The alternaters arbitrarily choose their first response, either a lie or the truth, but each subsequent statement has the opposite truth value from its predecessor. The principal asked everyone the same three questions in this order.

"Are you a truth-teller?" The principal gave a piece of candy to each of the 22 children who answered yes.

"Are you an alternater?" The principal gave a piece of candy to each of the 15 children who answered yes.

"Are you a liar?" The principal gave a piece of candy to each of the 9 children who answered yes.

How many pieces of candy in all did the principal give to the children who always tell the truth?

A 7

B 12

C 21

D 27

E 31

Solution:

For the first question, the truth-tellers will respond yes, the liars will respond yes, and the alternaters who decided to lie first will say yes. The alternaters who decide to tell the truth first will say no. Denote this as

$$22 = t + l + a_l.$$

For the second questions, the liars will respond yes, and the alternaters who decided to lie first will say yes (they are forced to tell the truth for this question). The truth-tellers will respond no, and the alternaters who told the truth first would lie this round,

responding no. Denote this as

$$15 = l + a_l.$$

From this, we get that $t = 7$. The principal only gives candy to children who always tell the truth in the first round, therefore only giving them 7 candies total.

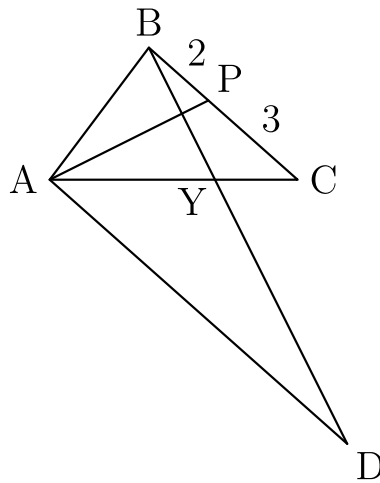
Thus, **A** is the correct answer.

13. Let $\triangle ABC$ be a scalene triangle. Point P lies on $\overline{BC}$ so that $\overline{AP}$ bisects $\angle BAC$. The line through B perpendicular to $\overline{AP}$ intersects the line through A parallel to $\overline{BC}$ at point D . Suppose $BP = 2$ and $PC = 3$. What is AD ?

- ☐ A 8
- ☐ B 9
- ☒ C 10
- ☐ D 11
- ☐ E 12

Solution:

Consider the following diagram:



Let Y be the intersection of $\overline{BD}$ and $\overline{AC}$. By the Angle Bisector Theorem, $AB : AC = BP : PC = 2 : 3$, so write $AB = 2x$ and $AC = 3x$.

Reflection across the angle bisector $\overline{AP}$ sends ray AB to ray AC . Because $BY \perp AP$, it sends B to Y . Thus $AY = AB = 2x$, and hence $YC = AC - AY = x$.

Since $AD \parallel BC$, with B, Y, D collinear and A, Y, C collinear, we have $\triangle BYC \sim \triangle DY A$. Therefore

$$\frac{AD}{BC} = \frac{AY}{YC} = 2.$$

Finally, $BC = BP + PC = 5$, so $AD = 2BC = 10$.

Thus, **C** is the correct answer.

- 14.** How many ways are there to split the integers 1 through 14 into 7 pairs such that in each pair, the greater number is at least 2 times the lesser number?

- ☐ A 108
- ☐ B 120
- ☐ C 126
- ☐ D 132
- ☒ E 144

Solution:

The numbers from 8 through 14 cannot be paired with one another, so they must be paired with the numbers from 1 through 7. In particular, 7 must be paired with 14, since no other available number is at least twice 7.

Now let's look at what the other numbers can pair with. 8 and 9 can pair with any number 1 – 4. 10 and 11 can pair with any number 1 – 5, and 12 and 13 can pair with any number 1 – 6.

8 can pair with 4 numbers, but then 9 only has 3 options since 8 took one. 10 then has 3 options, since 2 choices are taken, but it has one more to choose from (5). 11 then has 2 options, 12 has 2 options, and 13 only has 1.

Multiplying these together yields

$$4 \cdot 3 \cdot 3 \cdot 2 \cdot 2 = 144.$$

Thus, **E** is the correct answer.

15. Quadrilateral $ABCD$ with side lengths $AB = 7$, $BC = 24$, $CD = 20$, $DA = 15$ is inscribed in a circle. The area interior to the circle but exterior to the quadrilateral can be written in the form $\frac{a\pi - b}{c}$, where a , b , and c are positive integers such that a and c have no common prime factor. What is $a + b + c$?

A 260

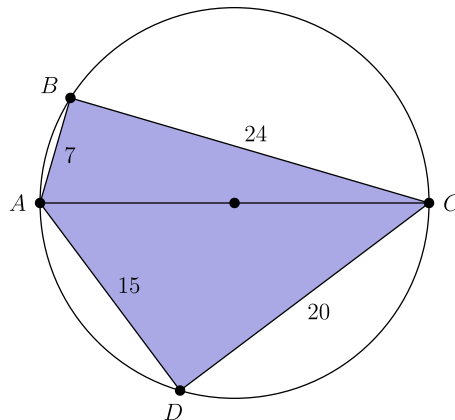
B 855

C 1235

D 1565

E 1997

Solution:



Notice that $7^2 + 24^2$ and $15^2 + 20^2$ are both the same. This forces $AC = 25$ since otherwise $\angle B$ and $\angle D$ would both be acute or obtuse, violating the fact that their sum is 180° .

Also since $\angle B$ is right, we know that AC is the diameter of the circle. The area of the circle is then $\frac{625}{4}\pi$.

To find the area of the quadrilateral, we can find the area of each of the triangles, which is

$$\frac{1}{2}(7 \cdot 24 + 20 \cdot 15) =$$

$$84 + 150 = 234.$$

To find the area outside the quadrilateral, we subtract to get

$$\frac{625}{4}\pi - 234 = \frac{625\pi - 936}{4}.$$

Therefore,

$$\begin{aligned} a + b + c &= 625 + 936 + 4 \\ &= 1565. \end{aligned}$$

Thus, **D** is the correct answer.

16. The roots of the polynomial $10x^3 - 39x^2 + 29x - 6$ are the height, length, and width of a rectangular box (right rectangular prism). A new rectangular box is formed by lengthening each edge of the original box by 2 units. What is the volume of the new box?

A $\frac{24}{5}$

B $\frac{42}{5}$

C $\frac{81}{5}$

D 30

E 48

Solution:

Let h , l , and w be the dimensions of the old box. Then the volume of the new box is

$$(h + 2)(l + 2)(w + 2).$$

Expanding, we get

$$\begin{aligned} hlw + 2(hl + hw + lw) \\ + 4(l + h + w) + 8. \end{aligned}$$

We can use Vieta's formulas to find the terms in this expression. We get that

$$\begin{aligned} hlw &= -\frac{D}{A} = \frac{3}{5}, \\ hl + hw + lw &= \frac{C}{A} = \frac{29}{10}, \end{aligned}$$

and

$$l + h + w = -\frac{B}{A} = \frac{39}{10}.$$

Plugging these values into the expression, we get

$$\frac{3}{5} + 2 \cdot \frac{29}{10} + 4 \cdot \frac{39}{10} + 8 = 30.$$

Thus, **D** is the correct answer.

17. How many three-digit positive integers $\underline{a} \underline{b} \underline{c}$ are there whose nonzero digits a, b , and c satisfy

$$0.\overline{\underline{a} \underline{b} \underline{c}} = \frac{1}{3}(0.\overline{a} + 0.\overline{b} + 0.\overline{c})?$$

(The bar indicates repetition, thus $0.\overline{\underline{a} \underline{b} \underline{c}}$ is the infinite repeating decimal $0.\underline{a} \underline{b} \underline{c} \underline{a} \underline{b} \underline{c} \dots$)

- A 9
- B 10
- C 11
- D 13**
- E 14

Solution:

The repeating decimals satisfy

$$\begin{aligned} 0.\overline{\underline{a} \underline{b} \underline{c}} &= \frac{100a + 10b + c}{999}, \\ 0.\overline{a} &= \frac{a}{9}, \quad 0.\overline{b} = \frac{b}{9}, \\ 0.\overline{c} &= \frac{c}{9}. \end{aligned}$$

Substitution and multiplication by 999 give

$$100a + 10b + c = 37(a + b + c),$$

or

$$7a = 3b + 4c.$$

For each $a \in \{1, 2, \dots, 9\}$, checking the nonzero digits b, c in this linear equation gives

$(1, 1, 1), (2, 2, 2), (3, 3, 3),$
 $(4, 4, 4), (4, 8, 1), (5, 1, 8),$
 $(5, 5, 5), (5, 9, 2), (6, 2, 9),$
 $(6, 6, 6), (7, 7, 7), (8, 8, 8),$
 $(9, 9, 9).$

Thus there are **13** integers.

Thus, **D** is the correct solution.

18. Let T_k be the transformation of the coordinate plane that first rotates the plane k degrees counterclockwise around the origin and then reflects the plane across the y -axis. What is the least positive integer n such that performing the sequence of transformations $T_1, T_2, T_3, \dots, T_n$ returns the point $(1, 0)$ back to itself?

A 359

B 360

C 719

D 720

E 721

Solution:

Since we are working with angles and reflections, working with polar coordinates would make this problem easier to deal with.

Let (r, θ) be a polar coordinate. Rotating this by k degrees counterclockwise maps the point to $(r, \theta + k^\circ)$ and then reflecting it maps it to $(r, 180^\circ - \theta - k^\circ)$.

Therefore, we have that

$$T_k(r, \theta) = (r, 180^\circ - \theta - k^\circ).$$

From this, we can see that

$$\begin{aligned} T_{k+1}(T_k(r, \theta)) &= \\ T_{k+1}(r, 180^\circ - \theta - k^\circ) &= \\ (r, \theta - 1^\circ). \end{aligned}$$

Now, let's analyze what happens to the point $(1, 0^\circ)$.

After T_1 , we get $(1, 179^\circ)$.

After T_2 , we get $(1, -1^\circ)$.

After T_3 , we get $(1, 178^\circ)$.

After T_4 , we get $(1, -2^\circ)$.

⋮

After T_{2m-1} , we get $(1, 180^\circ - m^\circ)$.

After T_{2m} , we get $(1, -m^\circ)$.

From this, we can see that the first time the angle is back to 0° is after $T_{2(180)-1} = T_{359}$. Therefore $n = 359$.

Thus, **A** is the correct answer.

19. Define L_n as the least common multiple of all the integers from 1 to n inclusive. There is a unique integer h such that

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{17} = \frac{h}{L_{17}}$$

What is the remainder when h is divided by 17?

- ☐ A 1
- ☐ B 3
- ☒ C 5
- ☐ D 7
- ☐ E 9

Solution:

Multiplying the harmonic sum by L_{17} , we get $h = \sum_{i=1}^{17} \frac{L_{17}}{i}$.

For $1 \leq i \leq 16$, the term $\frac{L_{17}}{i}$ is still divisible by 17, so these terms contribute 0 (mod 17).

Thus $h \equiv \frac{L_{17}}{17} \pmod{17}$. The least common multiple L_{17} contains the prime-power factors 16, 9, 5, 7, 11, 13, 17, so

$$\frac{L_{17}}{17} \equiv 16 \cdot 9 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \pmod{17}.$$

Reducing modulo 17, this is $(-1) \cdot 9 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \equiv 5 \pmod{17}$.

Thus, **C** is the correct answer.

20. A four-term sequence is formed by adding each term of a four-term arithmetic sequence of positive integers to the corresponding term of a four-term geometric sequence of positive integers. The first three terms of the resulting four-term sequence are 57, 60, and 91. What is the fourth term of this sequence?

A 190

B 194

C 198

D 202

E 206

Solution:

Let the arithmetic sequence be

$$a, a + d, a + 2d, a + 3d$$

and the geometric sequence be

$$b, br, br^2, br^3.$$

Then

$$a + b = 57, \tag{1}$$

$$a + d + br = 60, \tag{2}$$

and

$$a + 2d + br^2 = 91. \tag{3}$$

Subtracting (1) from (2) and (2) from (3), we get

$$d + b(r - 1) = 3$$

and

$$d + br(r - 1) = 31.$$

Subtracting these, we get

$$b(r - 1)^2 = 28.$$

Let $t = b(r - 1) = br - b$, which is an integer. Then $t^2 = 28b$, so $t = 14u$ and $b = 7u^2$ for some nonzero integer u . Because $a = 57 - b > 0$, we must have $u \in \{-2, -1, 1, 2\}$.

Now $r = 1 + \frac{t}{b} = 1 + \frac{2}{u}$. The cases $u = -1$ and $u = -2$ give $r = -1$ and $r = 0$, respectively, so they cannot produce a positive geometric sequence. If $u = 2$, then $b = 28$, $r = 2$, $a = 29$, and $d = 3 - t = -25$, making the third arithmetic term negative.

Therefore $u = 1$, so $b = 7$, $r = 3$, $a = 50$, and $d = -11$. The arithmetic sequence is

$$50, 39, 28, 17,$$

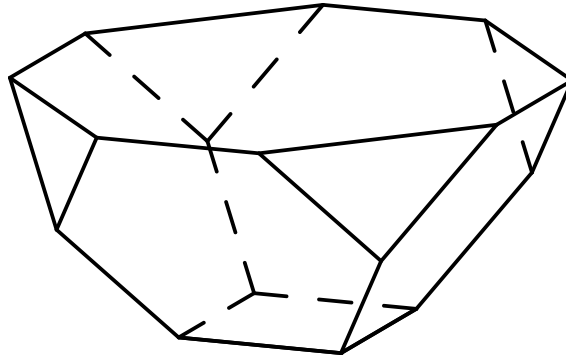
and the geometric sequence is

$$7, 21, 63, 189.$$

The desired answer is $17 + 189 = 206$.

Thus, **E** is the correct answer.

21. A bowl is formed by attaching four regular hexagons of side 1 to a square of side 1. The edges of the adjacent hexagons coincide, as shown in the figure. What is the area of the octagon obtained by joining the top eight vertices of the four hexagons, situated on the rim of the bowl?



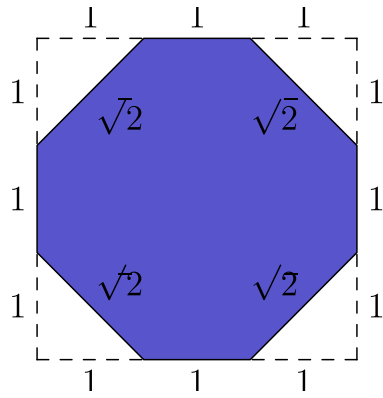
- ☐ A 6
- ☒ B 7
- ☐ C $5 + 2\sqrt{2}$
- ☐ D 8
- ☐ E 9

Solution:

View the rim from directly above. Place the bottom square at $(\pm\frac{1}{2}, \pm\frac{1}{2}, 0)$. In a regular hexagon, an edge adjacent to the attached side has components $\frac{1}{2}$ parallel and $\frac{\sqrt{3}}{2}$ perpendicular to that side.

At a corner of the bottom square, the corresponding edges of two adjacent hexagons coincide. Comparing their horizontal components shows that the horizontal component of a unit vector perpendicular to an attached side within its hexagon is $\frac{1}{\sqrt{3}}$. The opposite side of a regular hexagon is $\sqrt{3}$ units from the attached side, so its horizontal outward displacement is 1.

Consequently, the four unit-length sides of the rim lie one unit beyond the four sides of the bottom square. Its top view is therefore a 3-by-3 square with four isosceles right corner triangles of leg 1 removed:

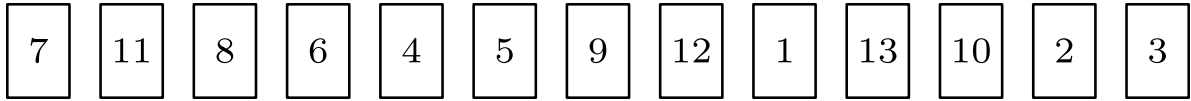


Its area is

$$3^2 - 4 \left(\frac{1}{2} \cdot 1^2 \right) = 7.$$

Thus, **B** is the correct answer.

22. Suppose that 13 cards numbered $1, 2, 3, \dots, 13$ are arranged in a row. The task is to pick them up in numerically increasing order, working repeatedly from left to right. In the example below, cards $1, 2, 3$ are picked up on the first pass, 4 and 5 on the second pass, 6 on the third pass, $7, 8, 9, 10$ on the fourth pass, and $11, 12, 13$ on the fifth pass. For how many of the $13!$ possible orderings of the cards will the 13 cards be picked up in exactly two passes?



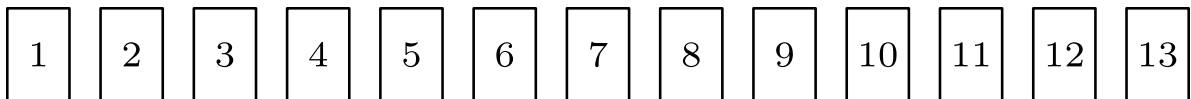
- A 4082
- B 4095
- C 4096
- D 8178**
- E 8191

Solution:

Let n be the number of cards picked up on the first pass, where $1 \leq n \leq 12$.

If we choose the spaces that the n cards occupy, the positions of the remaining cards are determined since they must be placed in order.

There are $\binom{13}{n}$ ways to choose where the n cards go, but if the n cards are placed at the very beginning, then all the cards will be picked up on the first pass.



Therefore, for a given n there are $\binom{13}{n} - 1$ ways to arrange the cards.

Summing over $1 \leq n \leq 12$, we get

$$\begin{aligned} & \sum_{n=1}^{12} \left(\binom{13}{n} - 1 \right) \\ &= (2^{13} - 2) - 12 \\ &= 8178. \end{aligned}$$

Thus, **D** is the correct answer.

23. Isosceles trapezoid $ABCD$ has parallel sides $\overline{AD}$ and $\overline{BC}$, with $BC < AD$ and $AB = CD$. There is a point P in the plane such that $PA = 1$, $PB = 2$, $PC = 3$, and $PD = 4$. What is $\frac{BC}{AD}$?

A $\frac{1}{4}$

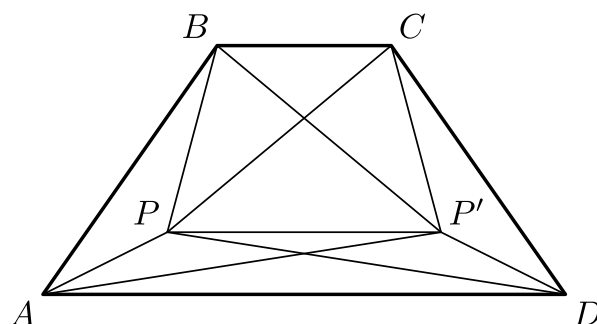
B $\frac{1}{3}$

C $\frac{1}{2}$

D $\frac{2}{3}$

E $\frac{3}{4}$

Solution:



Let P' be the reflection of P across the perpendicular bisector of $\overline{BC}$.

This forms two new isosceles trapezoids: $CBPP'$ and $DAPP'$.

Therefore, we get

$$P'A = PD = 4$$

$$P'D = PA = 1$$

$$P'C = PB = 2$$

$$P'B = PC = 3.$$

Using Ptolemy's theorem, we know that the product of the diagonals is equal to the sum of the products of the opposite sides. Therefore:

$$PP' \cdot AD + 1 = 16$$

$$PP' \cdot BC + 4 = 9.$$

This gets us $PP' \cdot AD = 15$ and $PP' \cdot BC = 5$. Dividing these two equations yields

$$\frac{BC}{AD} = \frac{1}{3}.$$

Thus, **B** is the correct answer.

24. How many strings of length 5 formed from the digits 0, 1, 2, 3, 4, are there such that for each $j \in \{1, 2, 3, 4\}$, at least j of the digits are less than j ?

(For example, 02214 satisfies this condition because it contains at least 1 digit less than 1, at least 2 digits less than 2, at least 3 digits less than 3, and at least 4 digits less than 4. The string 23404 does not satisfy the condition because it does not contain at least 2 digits less than 2.)

A 500

B 625

C 1089

D 1199

E 1296

Solution:

Regard the five digits, in order, as the preferred parking spaces of five cars. Spaces are numbered 0, 1, 2, 3, 4, and each car takes its preferred space if possible, or else the first empty space to its right. If the preferences sorted into nondecreasing order are $b_1 \leq b_2 \leq \dots \leq b_5$, all cars park exactly when

$$b_i \leq i - 1 \quad (1 \leq i \leq 5).$$

These inequalities are precisely the conditions in the problem.

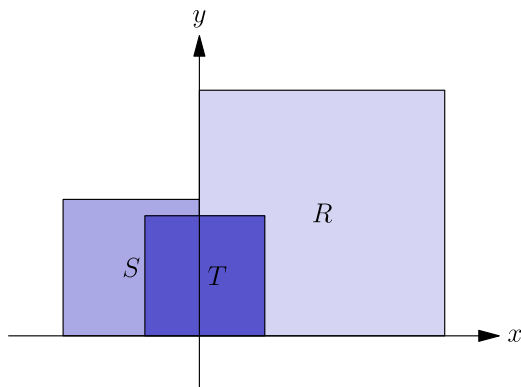
To count such preference strings, add a sixth space and arrange spaces 0, 1, ..., 5 in a circle. For any of the 6^5 preference strings, all five cars park and exactly one space remains empty. Rotating every preference by one position rotates the empty space as well. Thus each orbit of six preference strings has each possible empty space exactly once.

Therefore exactly $\frac{6^5}{6} = 6^4 = 1296$ circular preference strings leave space 5 empty. No car in such a string prefers space 5, and cutting the circle immediately after that empty space gives exactly a successful parking sequence on spaces 0 through 4. Hence the desired number of strings is 1296.

Thus, **E** is the correct answer.

25. Let R , S , and T be squares that have vertices at lattice points (i.e., points whose coordinates are both integers) in the coordinate plane, together with their interiors.

The bottom edge of each square is on the x -axis. The left edge of R and the right edge of S are on the y -axis, and R contains $\frac{9}{4}$ as many lattice points as does S . The top two vertices of T are in $R \cup S$, and T contains $\frac{1}{4}$ of the lattice points contained in $R \cup S$. See the figure (not drawn to scale).



The fraction of lattice points in S that are in $S \cap T$ is 27 times the fraction of lattice points in R that are in $R \cap T$. What is the minimum possible value of the edge length of R plus the edge length of S plus the edge length of T ?

A 336

B 337

C 338

D 339

E 340

Solution:

Let r be the number of lattice points on the side length of R . Similarly define s for S and t for T . Note that the number of lattice points in a rectangle is the product of the number of lattice points along its width and the number of lattice points along its length.

The first condition gives us that

$$\begin{aligned} r^2 &= \frac{9}{4} \cdot s^2 \\ r &= \frac{3}{2} \cdot s \end{aligned} \tag{1}$$

The number of lattice points in $R \cup S$ is the sum of the lattice points in each of the regions, but there is overlap along the y -axis where S touches it.

The second condition, therefore, yields

$$\begin{aligned} t^2 &= \frac{1}{4}(r^2 + s^2 - s) \\ t^2 &= \frac{1}{4}\left(\frac{9}{4} \cdot s^2 + s^2 - s\right) \\ t^2 &= \frac{1}{4} \cdot \frac{13s^2 - 4s}{4} \\ 16t^2 &= s(13s - 4). \end{aligned}$$

From (1), we get that s is a multiple of 2. We can substitute s with $2j$ to get

$$\begin{aligned} 16t^2 &= 2j(26j - 4) \\ 4t^2 &= j(13j - 2). \end{aligned}$$

For the product to be divisible by 4, j must be divisible by 2. We can again substitute j with $2k$ to get

$$\begin{aligned} 4t^2 &= 2k(26k - 2) \\ t^2 &= k(13k - 1) \end{aligned} \tag{2}$$

Let x be the number of lattice points along the bottom of the rectangle formed by $S \cap T$ and y be the number of lattice points along the bottom of the rectangle formed by $R \cap T$.

Using these variables, we get that the number of lattice points in $S \cap T$ is xt and in $R \cap T$ is yt .

The third condition gives us that

$$\begin{aligned}\frac{xt}{s^2} &= 27 \cdot \frac{yt}{r^2} \\ \frac{x}{s^2} &= 27 \cdot \frac{y}{\frac{9}{4}s^2} \\ x &= 12y.\end{aligned}$$

We also know that $t = x + y - 1$ (accounting for overlap), and this yields

$$t = 13y - 1 \tag{3}$$

(3) gives us that

$$\begin{aligned}t &\equiv -1 \pmod{13}, \\ t^2 &\equiv 1 \pmod{13}.\end{aligned}$$

However, by (2), we get that

$$\begin{aligned}t^2 &\equiv -k \pmod{13} \\ k &\equiv -1 \pmod{13}.\end{aligned}$$

By (2), we also get that k is a perfect square since it is relatively prime to $13k - 1$, and they must multiply to a perfect square.

Thus k must be a perfect square satisfying $k \equiv -1 \pmod{13}$. The smaller positive squares 1, 4, 9, 16 do not satisfy this congruence, while $k = 25$ does, so 25 is the least possible value.

From this value of k , we get that $j = 2 \cdot 25 = 50$, $s = 2 \cdot 50 = 100$, and $r = \frac{3}{2} \cdot 100 = 150$. We can also find that

$$\begin{aligned}t^2 &= 25(13 \cdot 25 - 1) = 25 \cdot 324 \\ t &= 5 \cdot 18 = 90.\end{aligned}$$

Therefore,

$$r + s + t = 340.$$

The question, however, asked for the sum of the side lengths. The side lengths of the

squares are 1 less than the number of lattice points on the side, so we have to subtract 3.

This value is attainable: equation (3) gives $y = 7$ and hence $x = 84$. Since $x \leq s = 100$ and $y \leq r = 150$, a square T with 90 lattice points per side can straddle the y -axis with the required overlaps, and its top vertices lie in $R \cup S$.

Therefore, the desired answer is $340 - 3 = 337$.

Thus, **B** is the correct answer.

Problems: <https://live.poshenloh.com/past-contests/amc10/2022A>

