

2021 AMC 10B Spring Solutions

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1. How many integer values of x satisfy $|x| < 3\pi$?

- ☐ A 9
- ☐ B 10
- ☐ C 18
- ☒ D 19
- ☐ E 20

Solution:

Every integer from -9 to 9 , inclusive, works. This yields $9 - (-9) + 1 = 19$ solutions.

Thus, the correct answer is **D**.

2. What is the value of

$$\sqrt{(3 - 2\sqrt{3})^2} + \sqrt{(3 + 2\sqrt{3})^2}?$$

A 0

B $4\sqrt{3} - 6$

C 6

D $4\sqrt{3}$

E $4\sqrt{3} + 6$

Solution:

Because $\sqrt{u^2} = |u|$, the expression equals

$$|3 - 2\sqrt{3}| + |3 + 2\sqrt{3}|.$$

Since $2\sqrt{3} > 3$, this becomes

$$(2\sqrt{3} - 3) + (3 + 2\sqrt{3}) = 4\sqrt{3}.$$

Thus, the correct answer is **D**.

3. In an after-school program for juniors and seniors, there is a debate team with an equal number of students from each class on the team. Among the 28 students in the program, 25% of the juniors and 10% of the seniors are on the debate team. How many juniors are in the program?

- A 5
- B 6
- C 8**
- D 11
- E 20

Solution:

Let the number of juniors be j and the number of seniors be s . Then, $j + s = 28$ and $0.25j = 0.1s$. This means $2.5j = s$, so $3.5j = 28$. This makes $j = 8$.

Thus, the correct answer is **C**.

4. At a math contest, 57 students are wearing blue shirts, and another 75 students are wearing yellow shirts. The 132 students are assigned into 66 pairs. In exactly 23 of these pairs, both students are wearing blue shirts. In how many pairs are both students wearing yellow shirts?

A 23

B 32

C 37

D 41

E 64

Solution:

There are $2 \cdot 23 = 46$ students with blue shirts that are in a pair with just blue shirts. This means there are $57 - 46 = 11$ students in blue shirts who are paired with someone wearing a yellow shirt, meaning exactly 11 people wearing yellow shirts are paired with someone wearing a blue shirt.

This leaves just 64 students wearing a yellow shirt who are paired with someone else wearing a yellow shirt. This yields $\frac{64}{2} = 32$ pairs.

Thus, the answer is **B**.

5. The ages of Jonie's four cousins are distinct single-digit positive integers. Two of the cousins' ages multiplied together give 24, while the other two multiply to 30. What is the sum of the ages of Jonie's four cousins?

A 21

B 22

C 23

D 24

E 25

Solution:

The only pair of single-digit positive integers with product 30 is 5, 6. The single-digit factor pairs of 24 are 3, 8 and 4, 6. Because all four ages are distinct, the pair 4, 6 would repeat the age 6, so the other pair must be 3, 8. Thus the ages are 3, 5, 6, 8, whose sum is 22.

Thus, the answer is **B**.

6. Ms. Blackwell gives an exam to two classes. The mean of the scores of the students in the morning class is 84, and the afternoon class's mean score is 70. The ratio of the number of students in the morning class to the number of students in the afternoon class is $\frac{3}{4}$. What is the mean of the scores of all the students?

A 74

B 75

C 76

D 77

E 78

Solution:

Let the number of people in the first class be $3x$. This means the number of people in the second class is $4x$.

Thus, the sum of the scores of the first class is $84 \cdot 3x = 252x$ and the sum of the scores for the people in the second class is $70 \cdot 4x = 280x$. This means the total sum is $532x$, with $7x$ people.

Therefore, the average of all the students is $\frac{532x}{7x} = 76$.

Thus, the correct answer is **C**.

7. In a plane, four circles with radii 1, 3, 5, and 7 are tangent to line ℓ at the same point A , but they may be on either side of ℓ . Region S consists of all the points that lie inside exactly one of the four circles. What is the maximum possible area of region S ?

A 24π

B 32π

C 64π

D 65π

E 84π

Solution:

On one side of ℓ , circles tangent at A are nested. For nested circles with radii $r_1 > r_2 > \dots$, the points inside exactly one of those circles have area $\pi(r_1^2 - r_2^2)$ if there are at least two circles; a third smaller nested circle does not count because its points are inside three circles, not exactly one.

To maximize the area, put the circle of radius 7 alone on one side, and put the circles of radii 5, 3, 1 on the other side. This gives

$$\begin{aligned} &7^2\pi + (5^2 - 3^2)\pi \\ &= 49\pi + 16\pi = 65\pi. \end{aligned}$$

Thus, the answer is **D**.

8. Mr. Zhou places all the integers from 1 to 225 into a 15 by 15 grid. He places 1 in the middle square (eighth row and eighth column) and places other numbers one by one clockwise, as shown in part in the diagram below. What is the sum of the greatest number and the least number that appear in the second row from the top?

...	...	...	...	...	...	...
...	21	22	23	24	25	...
...	20	7	8	9	10	...
...	19	6	1	2	11	...
...	18	5	4	3	12	...
...	17	16	15	14	13	...
...	...	...	...	...	...	...

A 367

B 368

C 369

D 379

E 380

Solution:

In the outer 15×15 ring, the top row contains 211, 212, ..., 225, so the number just below 211 in the second row is 210. This is the greatest number in the second row.

The inner 13×13 spiral has $13^2 = 169$ in its upper-right corner. In the second row of the full grid, the inner-ring entries run from 157 to 169. Thus the least entry in that row is 157.

The required sum is $210 + 157 = 367$.

Thus, the answer is **A**.

9. The point $P(a, b)$ in the xy -plane is first rotated counterclockwise by 90° around the point $(1, 5)$ and then reflected about the line $y = -x$. The image of P after these two transformations is at $(-6, 3)$. What is $b - a$?

A 1

B 3

C 5

D 7

E 9

Solution:

Work backward. Reflecting $(-6, 3)$ across $y = -x$ gives $(-3, 6)$.

Now undo the 90° counterclockwise rotation by rotating $(-3, 6)$ clockwise about $(1, 5)$. Relative to $(1, 5)$, the point is $(-4, 1)$. A clockwise quarter-turn sends this to $(1, 4)$, and translating back gives $(2, 9)$.

Thus $a = 2$, $b = 9$, and $b - a = 7$.

Thus, the answer is **D**.

10. An inverted cone with base radius 12cm and height 18cm is full of water. The water is poured into a tall cylinder whose horizontal base has a radius of 24cm. What is the height in centimeters of the water in the cylinder?

A 1.5

B 3

C 4

D 4.5

E 6

Solution:

The volumes must be the same since the water is poured from one to another. The volume of the cone is $\frac{r^2 h \pi}{3} = \frac{12^2 \cdot 18 \pi}{3} = 864 \pi$.

The volume of the cylinder is $r^2 h \pi = 24^2 h \pi = 576 h \pi$. This makes $576 h \pi = 864 \pi$, so $h = 1.5$.

Thus, the answer is **A**.

11. Grandma has just finished baking a large rectangular pan of brownies. She is planning to make rectangular pieces of equal size and shape, with straight cuts parallel to the sides of the pan. Each cut must be made entirely across the pan. Grandma wants to make the same number of interior pieces as pieces along the perimeter of the pan. What is the greatest possible number of brownies she can produce?

A 24

B 30

C 48

D 60

E 64

Solution:

Suppose the cuts make an $l \times w$ grid of pieces. The number of interior pieces is $(l - 2)(w - 2)$, and the total number of pieces is lw . Since the number of interior pieces equals the number of perimeter pieces, the interior pieces make up half the total:

$$(l - 2)(w - 2) = \frac{lw}{2}.$$

Multiplying out gives $lw - 4l - 4w + 8 = 0$, or

$$(l - 4)(w - 4) = 8.$$

The positive factor pairs of 8 give $(l, w) = (5, 12)$ or $(6, 8)$, up to order. These produce 60 or 48 pieces, respectively, so the greatest possible number is 60.

Thus, the answer is **D**.

12. Let $N = 34 \cdot 34 \cdot 63 \cdot 270$. What is the ratio of the sum of the odd divisors of N to the sum of the even divisors of N ?

A 1 : 16

B 1 : 15

C 1 : 14

D 1 : 8

E 1 : 3

Solution:

Using prime factorization, we get

$$N = 2^3 \cdot 3^5 \cdot 5 \cdot 7 \cdot 17^2.$$

If we have an odd divisor x of N , then $2x, 4x, 8x$ are divisors of N , which has a combined sum of $14x$. If we take the sum of every odd divisor, then the even divisors must have a sum which is 14 times the sum of the odd divisors. Therefore, the requested ratio is 1 : 14.

Thus, the answer is **C**.

13. Let n be a positive integer and d be a digit such that the value of the numeral $\underline{32d}$ in base n equals 263 , and the value of the numeral $\underline{324}$ in base n equals the value of the numeral $\underline{11d1}$ in base six. What is $n + d$?

A 10

B 11

C 13

D 15

E 16

Solution:

The first statement means

$$3n^2 + 2n + d = 263.$$

Similarly, the second statement means

$$\begin{aligned} 3n^2 + 2n + 4 \\ &= 6^3 + 6^2 + 6d + 1 \\ &= 253 + 6d. \end{aligned}$$

Subtracting these shows us that

$$\begin{aligned} 4 - d &= 6d - 10 \\ 7d &= 14 \\ d &= 2. \end{aligned}$$

Therefore, $3n^2 + 2n + 2 = 263$, so $n(3n + 2) = 261$.

This implies $n = 9$. Therefore, $n + d = 11$.

Thus, the answer is **B**.

14. Three equally spaced parallel lines intersect a circle, creating three chords of lengths 38, 38, and 34. What is the distance between two adjacent parallel lines?

- A $5\frac{1}{2}$
- B 6**
- C $6\frac{1}{2}$
- D 7
- E $7\frac{1}{2}$

Solution:

The two chords of length 38 are equally far from the center of the circle. They cannot lie on the two outer lines: then the center would lie on the middle line, whose chord would be a diameter longer than 38, not the given 34. Therefore, the equal chords lie on adjacent lines, with the center halfway between them. Let that half-distance be d . Then each 38-chord is distance d from the center, and the 34-chord is distance $3d$ from the center.

If the circle has radius r , then

$$r^2 = 19^2 + d^2 = 17^2 + (3d)^2.$$

Thus $19^2 - 17^2 = 8d^2$, so $72 = 8d^2$, and $d = 3$. The distance between adjacent parallel lines is $2d = 6$.

Thus, the answer is **B**.

15. The real number x satisfies the equation

$$x + \frac{1}{x} = \sqrt{5}.$$

What is the value of $x^{11} - 7x^7 + x^3$?

A -1

B 0

C 1

D 2

E $\sqrt{5}$

Solution:

Since $x + \frac{1}{x} = \sqrt{5}$, squaring yields

$$x^2 + 2 + \frac{1}{x^2} = 5$$

$$x^2 + \frac{1}{x^2} = 3.$$

Squaring again yields

$$x^4 + 2 + \frac{1}{x^4} = 9$$

$$x^4 - 7 + \frac{1}{x^4} = 0.$$

Multiplying by x^7 yields

$$x^{11} - 7x^7 + x^3 = 0.$$

Thus, the answer is **B**.

16. Call a positive integer an uphill integer if every digit is strictly greater than the previous digit. For example, 1357, 89, and 5 are all uphill integers, but 32, 1240, and 466 are not. How many uphill integers are divisible by 15?

- A 4
- B 5
- C 6**
- D 7
- E 8

Solution:

If a number is divisible by 15, it has a units digit of 0 or 5. If the units digit is 0 and the digits are strictly increasing, then the number is 0, which isn't positive. Therefore, we can just look at numbers with a units digit of 5.

Next, we need to find uphill integers that are a multiple of 3. This means the other digits are a subset of {1, 2, 3, 4}. Taking the sum of the set must have a remainder of 1 when divided by 3. Also, having or taking out 3 wouldn't affect the remainder, so we can take the number of subsets without a 3 and multiply it by 2. There are only 3 such subsets, namely {1}, {4}, and {1, 2, 4}. Thus, there are 6 total subsets.

Thus, the correct answer is **C**.

17. Ravon, Oscar, Aditi, Tyrone, and Kim play a card game. Each person is given 2 cards out of a set of 10 cards numbered 1, 2, 3, ..., 10. The score of a player is the sum of the numbers of their cards. The scores of the players are as follows: Ravon--11, Oscar--4, Aditi--7, Tyrone--16, Kim--17. Which of the following statements is true?

A Ravon was given card 3.

B Aditi was given card 3.

C Ravon was given card 4.

D Aditi was given card 4.

E Tyrone was given card 7.

Solution:

Oscar's score of 4 forces his cards to be 1 and 3. From the remaining cards, the only pair with sum 7 is 2 and 5, so those are Aditi's cards.

The unused cards are now 4, 6, 7, 8, 9, 10. Tyrone's score of 16 can come only from $6 + 10$ or $7 + 9$. If Tyrone had 7 and 9, then the remaining cards 4, 6, 8, 10 could not be paired to give Ravon's score 11. Thus Tyrone has 6 and 10.

The remaining cards are 4, 7, 8, 9. Ravon must have 4 and 7 to score 11, leaving 8 and 9 for Kim's score of 17. Therefore, Ravon was given card 4.

Thus, the answer is **C**.

18. A fair 6-sided die is repeatedly rolled until an odd number appears. What is the probability that every even number appears at least once before the first occurrence of an odd number?

A $\frac{1}{120}$

B $\frac{1}{32}$

C $\frac{1}{20}$

D $\frac{3}{20}$

E $\frac{1}{6}$

Solution:

The probability that the first number is even is $\frac{3}{6}$.

The probability that the second distinct number is even is $\frac{2}{5}$.

The probability that the third distinct number is even is $\frac{1}{4}$.

The combined probability is

$$\frac{3 \cdot 2 \cdot 1}{6 \cdot 5 \cdot 4} = \frac{1}{20}.$$

Thus, the answer is **C**.

19. Suppose that S is a finite set of positive integers.

If the greatest integer in S is removed from S , then the average value (arithmetic mean) of the integers remaining is 32. If the least integer in S is also removed, then the average value of the integers remaining is 35. If the greatest integer is then returned to the set, the average value of the integers rises to 40. The greatest integer in the original set S is 72 greater than the least integer in S .

What is the average value of all the integers in the set S ?

A 36.2

B 36.4

C 36.6

D 36.8

E 37

Solution:

Let the sum of all the integers be s , the greatest number be g , the least number be l , and the size of S be n .

The two averages of sets with $n - 1$ elements give

$$s - l = 40(n - 1),$$

$$s - g = 32(n - 1).$$

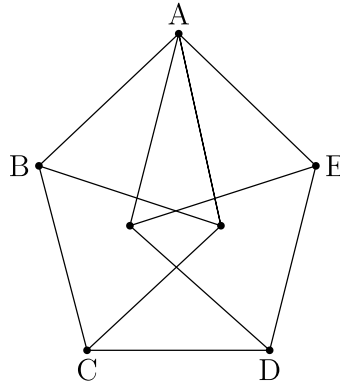
Subtracting and using $g - l = 72$ gives $72 = 8(n - 1)$, so $n = 10$.

Removing both extremes leaves 8 numbers with sum $8 \cdot 35 = 280$. On the other hand, removing only the least number leaves a sum of $9 \cdot 40 = 360$, so $g = 360 - 280 = 80$. Hence $l = 80 - 72 = 8$, and the original sum is $s = 280 + 80 + 8 = 368$.

The required average is $\frac{s}{n} = \frac{368}{10} = 36.8$.

Thus, the answer is **D**.

20. The figure below is constructed from 11 line segments, each of which has length 2. The area of pentagon $ABCDE$ can be written as $\sqrt{m} + \sqrt{n}$, where m and n are positive integers. What is $m + n$?



- A 20
- B 21
- C 22
- D 23**
- E 24

Solution:

Let F be the unlabeled point joined to A , B , and C . Because all the drawn segments have length 2, triangles ABF and CBF are equilateral and lie on opposite sides of BF . Hence $\angle ABC = 120^\circ$, so

$$[ABC] = \frac{1}{2} \cdot 2 \cdot 2 \sin 120^\circ = \sqrt{3}.$$

The same reasoning on the other side gives $[ADE] = \sqrt{3}$. Also, the Law of Cosines in $\triangle ABC$ gives $AC^2 = 12$, and similarly $AD^2 = 12$. Thus the altitude of isosceles triangle ACD to its base $CD = 2$ is

$$\sqrt{(\sqrt{12})^2 - 1^2} = \sqrt{11}.$$

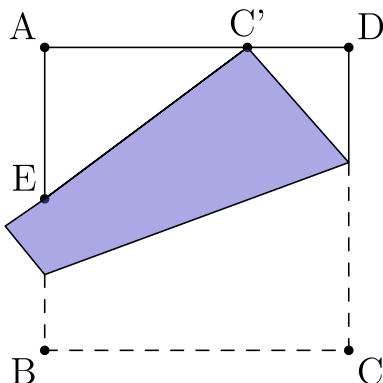
Therefore $[ACD] = \frac{1}{2} \cdot 2 \cdot \sqrt{11} = \sqrt{11}$. The pentagon's total area is

$$\sqrt{12} + \sqrt{11},$$

so $m + n = 12 + 11 = 23$.

Thus, the answer is **D**.

21. A square piece of paper has side length 1 and vertices A, B, C , and D in that order. As shown in the figure, the paper is folded so that vertex C meets edge $\overline{AD}$ at point C' , and edge $\overline{BC}$ intersects edge $\overline{AB}$ at point E . Suppose that $C'D = \frac{1}{3}$. What is the perimeter of $\triangle AEC'$?



A 2

B $1 + \frac{2}{3}\sqrt{3}$

C $\frac{13}{6}$

D $1 + \frac{3}{4}\sqrt{3}$

E $\frac{7}{3}$

Solution:

Use coordinates with $A = (0, 1)$, $B = (0, 0)$, $C = (1, 0)$, and $D = (1, 1)$. Since $C'D = \frac{1}{3}$, we have $C' = (\frac{2}{3}, 1)$, so $AC' = \frac{2}{3}$.

The fold reflects C to C' , so the image of side BC is the line through C' and E . Reflecting $B = (0, 0)$ across the perpendicular bisector of CC' gives $(-\frac{2}{15}, \frac{2}{5})$. The line through this point and C' meets AB at $E = (0, \frac{1}{2})$.

Thus $AE = \frac{1}{2}$, and

$$EC' = \sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{5}{6}.$$

The perimeter of $\triangle AEC'$ is

$$\frac{1}{2} + \frac{2}{3} + \frac{5}{6} = 2.$$

Thus, the answer is **A**.

22. Ang, Ben, and Jasmin each have 5 blocks, colored red, blue, yellow, white, and green; and there are 5 empty boxes. Each of the people randomly and independently of the other two people places one of their blocks into each box. The probability that at least one box receives 3 blocks all of the same color is $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n$?

A 47

B 94

C 227

D 471

E 542

Solution:

Fix Ang's placement and label each box by the color Ang put in it. Ben and Jasmin each choose a permutation of the five colors, so there are $(5!)^2$ equally likely pairs of placements.

For a specified set of k boxes to receive three blocks of the same color, both Ben and Jasmin must match Ang in those k boxes. This can happen in $((5 - k)!)^2$ ways. By inclusion-exclusion, the number of successful placement pairs is

$$\begin{aligned} & \binom{5}{1}(4!)^2 - \binom{5}{2}(3!)^2 \\ & + \binom{5}{3}(2!)^2 - \binom{5}{4}(1!)^2 \\ & + \binom{5}{5}(0!)^2. \end{aligned}$$

This equals

$$2880 - 360 + 40 - 5 + 1 = 2556.$$

Therefore the probability is

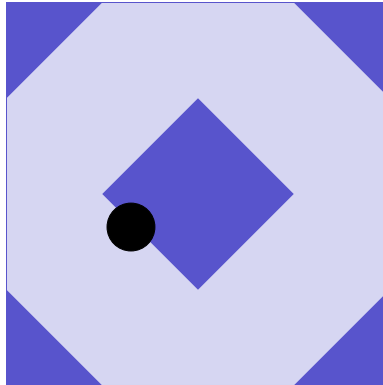
$$\frac{2556}{(5!)^2} = \frac{2556}{14400} = \frac{71}{400}.$$

Thus $m + n = 71 + 400 = 471$.

Thus, the answer is **D**.

23. A square with side length 8 is colored white except for 4 black isosceles right triangular regions with legs of length 2 in each corner of the square and a black diamond with side length $2\sqrt{2}$ in the center of the square, as shown in the diagram.

A circular coin with diameter 1 is dropped onto the square and lands in a random location where the coin is completely contained within the square. The probability that the coin will cover part of the black region of the square can be written as $\frac{1}{196} (a + b\sqrt{2} + \pi)$, where a and b are positive integers. What is $a + b$?



- A 64
- B 66
- C 68**
- D 70
- E 72

Solution:

The coin has radius $\frac{1}{2}$, so its center is uniformly distributed over a 7×7 square of area 49.

A shaded corner triangle contributes the set of center positions within distance $\frac{1}{2}$ of that triangle, inside the allowed center square. For each corner this is a right isosceles triangle whose altitude is $\frac{1+\sqrt{2}}{2}$, so its area is

$$\left(\frac{1 + \sqrt{2}}{2} \right)^2 = \frac{3 + 2\sqrt{2}}{4}.$$

All four corners contribute $3 + 2\sqrt{2}$.

The center shaded diamond is a square of side $2\sqrt{2}$. Expanding it by distance $\frac{1}{2}$ adds four rectangles of total area $4\sqrt{2}$ and four quarter-circles of total area $\frac{\pi}{4}$, in addition to the diamond's area 8. Thus the center contribution is

$$8 + 4\sqrt{2} + \frac{\pi}{4}.$$

The favorable area is

$$\begin{aligned} 3 + 2\sqrt{2} + 8 + 4\sqrt{2} + \frac{\pi}{4} \\ = 11 + 6\sqrt{2} + \frac{\pi}{4}. \end{aligned}$$

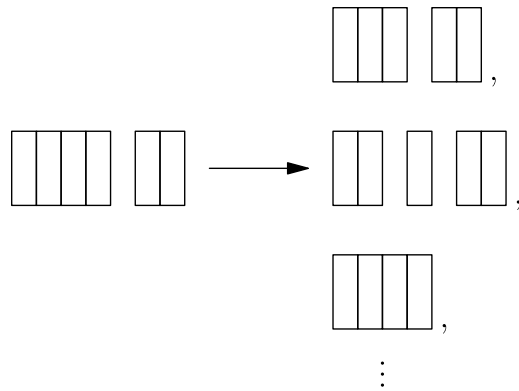
The probability is

$$\begin{aligned} \frac{11 + 6\sqrt{2} + \frac{\pi}{4}}{49} \\ = \frac{44 + 24\sqrt{2} + \pi}{196}. \end{aligned}$$

So $a + b = 44 + 24 = 68$.

Thus, the answer is **C**.

24. Arjun and Beth play a game in which they take turns removing one brick or two adjacent bricks from one “wall” among a set of several walls of bricks, with gaps possibly creating new walls. The walls are one brick tall. For example, a set of walls of sizes 4 and 2 can be changed into any of the following by one move: (3, 2), (2, 1, 2), (4), (4, 1), (2, 2), or (1, 1, 2).



Arjun plays first, and the player who removes the last brick wins. For which starting configuration is there a strategy that guarantees a win for Beth?

A (6, 1, 1)

B (6, 2, 1)

C (6, 2, 2)

D (6, 3, 1)

E (6, 3, 2)

Solution:

For a single wall of length n , compute its Sprague–Grundy value from the possible moves. For the wall lengths needed here, the values are

$$\begin{aligned} g(1) &= 1, & g(2) &= 2, \\ g(3) &= 3, & g(4) &= 1, \\ g(5) &= 4, & g(6) &= 3. \end{aligned}$$

For several walls, the position is losing for the player to move exactly when the xor of the wall values is 0. Evaluating the choices gives

$$(6, 1, 1) : 3 \oplus 1 \oplus 1 = 3,$$

$$(6, 2, 1) : 3 \oplus 2 \oplus 1 = 0,$$

$$(6, 2, 2) : 3 \oplus 2 \oplus 2 = 3,$$

$$(6, 3, 1) : 3 \oplus 3 \oplus 1 = 1,$$

$$(6, 3, 2) : 3 \oplus 3 \oplus 2 = 2.$$

Only $(6, 2, 1)$ is losing for the player to move, so Beth has a guaranteed win exactly for that starting configuration.

Thus, the answer is **B**.

25. Let S be the set of lattice points in the coordinate plane, both of whose coordinates are integers between 1 and 30, inclusive. Exactly 300 points in S lie on or below a line with equation $y = mx$. The possible values of m lie in an interval of length $\frac{a}{b}$, where a and b are relatively prime positive integers. What is $a + b$?

A 31

B 47

C 62

D 72

E 85

Solution:

For a fixed slope m , the number of points in S on or below $y = mx$ is

$$\sum_{x=1}^{30} \lfloor mx \rfloor,$$

for the slopes near the answer.

At $m = \frac{2}{3}$, grouping $x = 3k + 1, 3k + 2, 3k + 3$ for $k = 0, 1, \dots, 9$ gives

$$\left\lfloor \frac{2x}{3} \right\rfloor = 2k, 2k + 1, 2k + 2,$$

whose sum over each block is $6k + 3$. Thus the total is

$$\sum_{k=0}^9 (6k + 3) = 270 + 30 = 300.$$

If $m < \frac{2}{3}$, the ten points with ratios $\frac{y}{x} = \frac{2}{3}$ are no longer counted, so the count is less than 300. Therefore the lower end is $\frac{2}{3}$.

The next possible ratio $\frac{y}{x}$ greater than $\frac{2}{3}$, with $1 \leq x, y \leq 30$, is minimized by checking x modulo 3. The best candidates are

$$\frac{19}{28}, \quad \frac{20}{29}, \quad \frac{21}{30} = \frac{7}{10},$$

and the smallest is $\frac{19}{28}$. Hence the interval length is

$$\frac{19}{28} - \frac{2}{3} = \frac{1}{84}.$$

Thus $a + b = 1 + 84 = 85$.

Thus, the answer is **E**.

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