

2020 AMC 10A Solutions

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1. What value of x satisfies

$$x - \frac{3}{4} = \frac{5}{12} - \frac{1}{3}?$$

A $-\frac{2}{3}$

B $\frac{7}{36}$

C $\frac{7}{12}$

D $\frac{2}{3}$

E $\frac{5}{6}$

Solution:

The right side is $\frac{5}{12} - \frac{1}{3} = \frac{5}{12} - \frac{4}{12} = \frac{1}{12}$. Thus $x = \frac{3}{4} + \frac{1}{12} = \frac{9}{12} + \frac{1}{12} = \frac{5}{6}$.

Thus, **E** is the correct answer.

2. The numbers 3, 5, 7, a , and b have an average (arithmetic mean) of 15. What is the average of a and b ?

- ☐ A 0
- ☐ B 15
- ☒ C 30
- ☐ D 45
- ☐ E 60

Solution:

The five numbers have total sum $5 \cdot 15 = 75$. Since $3 + 5 + 7 = 15$, we have $a + b = 75 - 15 = 60$, so the average of a and b is 30. Thus, **C** is the correct answer.

3. Assuming $a \neq 3$, $b \neq 4$, and $c \neq 5$, what is the value in simplest form of the following expression?

$$\frac{a-3}{5-c} \cdot \frac{b-4}{3-a} \cdot \frac{c-5}{4-b}$$

A -1

B 1

C $\frac{abc}{60}$

D $\frac{1}{abc} - \frac{1}{60}$

E $\frac{1}{60} - \frac{1}{abc}$

Solution:

Rewrite the denominator factors as $5 - c = -(c - 5)$, $3 - a = -(a - 3)$, and $4 - b = -(b - 4)$. The expression becomes $\frac{(a-3)(b-4)(c-5)}{-(a-3)(b-4)(c-5)} = -1$. Thus, **A** is the correct answer.

4. A driver travels for 2 hours at 60 miles per hour, during which her car gets 30 miles per gallon of gasoline. She is paid \$0.50 per mile, and her only expense is gasoline at \$2.00 per gallon. What is her net rate of pay, in dollars per hour, after this expense?

A 20

B 22

C 24

D 25

E 26

Solution:

The driver travels $2 \cdot 60 = 120$ miles, so she is paid $120 \cdot \$0.50 = \60 . The trip uses $\frac{120}{30} = 4$ gallons of gasoline, costing $4 \cdot \$2 = \8 . Her net pay is $60 - 8 = 52$ dollars over 2 hours, or 26 dollars per hour. Thus, **E** is the correct answer.

5. What is the sum of all real numbers x for which

$$|x^2 - 12x + 34| = 2?$$

- ☐ A 12
- ☐ B 15
- ☒ C 18
- ☐ D 21
- ☐ E 25

Solution:

The equation means $x^2 - 12x + 34 = 2$ or $x^2 - 12x + 34 = -2$. The first gives $x^2 - 12x + 32 = 0$, with roots 4 and 8. The second gives $(x - 6)^2 = 0$, with root 6. The sum of all real solutions is $4 + 8 + 6 = 18$. Thus, **C** is the correct answer.

6. How many 4-digit positive integers (that is, integers between 1000 and 9999, inclusive) having only even digits are divisible by 5?

A 80

B 100

C 125

D 200

E 500

Solution:

The last digit must be 0, because the number is divisible by 5 and all digits are even. The thousands digit can be 2, 4, 6, or 8, and each of the hundreds and tens digits has 5 choices. Thus there are $4 \cdot 5 \cdot 5 = 100$ such integers. Thus, **B** is the correct answer.

7. The 25 integers from -10 to 14 , inclusive, can be arranged to form a 5-by-5 square in which the sum of the numbers in each row, the sum of the numbers in each column, and the sum of the numbers along each of the main diagonals are all the same. What is the value of this common sum?

- ☐ A 2
- ☐ B 5
- ☒ C 10
- ☐ D 25
- ☐ E 50

Solution:

The sum of the integers from -10 to 14 is $25 \cdot \frac{-10 + 14}{2} = 50$. If every row has common sum S , then the five row sums add to 50, so $5S = 50$ and $S = 10$. Thus, **C** is the correct answer.

8. What is the value of

$$\begin{aligned} &1 + 2 + 3 - 4 + 5 + 6 + 7 - 8 \\ &\quad + \cdots \\ &\quad + 197 + 198 + 199 - 200? \end{aligned}$$

A 9,800

B 9,900

C 10,000

D 10,100

E 10,200

Solution:

Group the terms in blocks of four: $(1 + 2 + 3 - 4) + (5 + 6 + 7 - 8) + \cdots + (197 + 198 + 199 - 200)$. The j th block is $(4j - 3) + (4j - 2) + (4j - 1) - 4j = 8j - 6$.

There are 50 blocks, so the sum is $\sum_{j=1}^{50} (8j - 6) = 8 \cdot \frac{50 \cdot 51}{2} - 6 \cdot 50 = 9900$. Thus,

B is the correct answer.

9. A single bench section at a school event can hold either 7 adults or 11 children. When N bench sections are connected end to end, an equal number of adults and children seated together will occupy all the bench space. What is the least possible positive integer value of N ?

- A 9
- B 18**
- C 27
- D 36
- E 77

Solution:

If the equal number of adults and children is P , then the adults use $\frac{P}{7}$ bench sections and the children use $\frac{P}{11}$ bench sections. Thus $N = P \left(\frac{1}{7} + \frac{1}{11} \right) = \frac{18P}{77}$.

The least positive integer occurs when $P = 77$, giving $N = 18$. Thus, **B** is the correct answer.

10. Seven cubes, whose volumes are 1, 8, 27, 64, 125, 216, and 343 cubic units, are stacked vertically to form a tower in which the volumes of the cubes decrease from bottom to top. Except for the bottom cube, the bottom face of each cube lies completely on top of the cube below it. What is the total surface area of the tower (including the bottom) in square units?

A 644

B 658

C 664

D 720

E 749

Solution:

The cube side lengths are 1, 2, 3, 4, 5, 6, 7, stacked from largest on bottom to smallest on top. The sum of the surface areas of the separate cubes is $6(1^2 + 2^2 + \dots + 7^2) = 6 \cdot 140 = 840$.

Each contact hides two square faces, with areas $1^2, 2^2, \dots, 6^2$. Subtracting these hidden faces gives $840 - 2(1^2 + 2^2 + \dots + 6^2) = 840 - 182 = 658$. Thus, **B** is the correct answer.

11. What is the median of the following list of 4040 numbers?

$$1, 2, 3, \dots, 2020, \\ 1^2, 2^2, 3^2, \dots, 2020^2$$

A 1974.5

B 1975.5

C 1976.5

D 1977.5

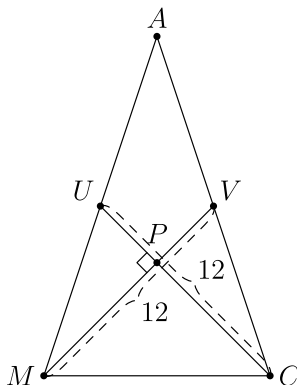
E 1978.5

Solution:

For a number near the median, the sorted list includes all ordinary integers up to that number and all squares up to that number. Since $44^2 = 1936$ and $45^2 = 2025$, there are 44 squares not exceeding any number from 1936 through 2020.

At 1975, there are $1975 + 44 = 2019$ list entries at most 1975. At 1976, there are 2020 entries at most 1976, so the 2020th entry is 1976, and the next is 1977. The median is 1976.5. Thus, C is the correct answer.

12. Triangle AMC is isosceles with $AM = AC$. Medians $\overline{MV}$ and $\overline{CU}$ are perpendicular to each other, and $MV = CU = 12$. What is the area of $\triangle AMC$?



- A 48
- B 72
- C 96**
- D 144
- E 192

Solution:

Let the centroid be the origin. Since a centroid divides each median in a $2 : 1$ ratio, we may place median MV horizontally with $M = (8, 0)$ and $V = (-4, 0)$, and median CU vertically with $C = (0, 8)$ and $U = (0, -4)$.

Because U is the midpoint of AM , we get $A = 2U - M = (-8, -8)$. The area of $\triangle AMC$ is $\frac{1}{2}|(16, 8) \times (8, 16)| = \frac{1}{2}(256 - 64) = 96$. Thus, **C** is the correct answer.

13. A frog sitting at the point $(1, 2)$ begins a sequence of jumps, where each jump is parallel to one of the coordinate axes and has length 1, and the direction of each jump (up, down, right, or left) is chosen independently at random. The sequence ends when the frog reaches a side of the square with vertices $(0, 0)$, $(0, 4)$, $(4, 4)$, and $(4, 0)$. What is the probability that the sequence of jumps ends on a vertical side of the square?

A $\frac{1}{2}$

B $\frac{5}{8}$

C $\frac{2}{3}$

D $\frac{3}{4}$

E $\frac{7}{8}$

Solution:

Let $p(x, y)$ be the probability of eventually hitting a vertical side first from point (x, y) . By symmetry, set $a = p(1, 1) = p(1, 3)$, $b = p(2, 1) = p(2, 3)$, $c = p(1, 2)$, and $d = p(2, 2)$.

The averaging equations are $a = \frac{1 + b + c}{4}$, $b = \frac{2a + d}{4}$, $c = \frac{1 + 2a + d}{4}$, and $d = \frac{b + c}{2}$. Solving gives $c = \frac{5}{8}$, which is the desired probability from $(1, 2)$. Thus, **B** is the correct answer.

14. Real numbers x and y satisfy $x + y = 4$ and $x \cdot y = -2$. What is the value of

$$x + \frac{x^3}{y^2} + \frac{y^3}{x^2} + y?$$

A 360

B 400

C 420

D 440

E 480

Solution:

Let $S_k = x^k + y^k$. Since $x + y = 4$ and $xy = -2$, the numbers x and y satisfy $t^2 - 4t - 2 = 0$, so $S_k = 4S_{k-1} + 2S_{k-2}$.

Using $S_0 = 2$ and $S_1 = 4$, we get $S_2 = 20$, $S_3 = 88$, $S_4 = 392$, and $S_5 = 1744$. The expression is $x + y + \frac{x^5 + y^5}{x^2y^2} = 4 + \frac{1744}{4} = 440$. Thus, **D** is the correct answer.

15. A positive integer divisor of $12!$ is chosen at random. The probability that the divisor chosen is a perfect square can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n$?

A 3

B 5

C 12

D 18

E 23

Solution:

The prime factorization of $12!$ is $2^{10}3^55^27^111^1$. Therefore $12!$ has $(10 + 1)(5 + 1) \cdot (2 + 1)(1 + 1)(1 + 1) = 792$ positive divisors.

A square divisor must use only even exponents, giving $6 \cdot 3 \cdot 2 \cdot 1 \cdot 1 = 36$ square divisors. The probability is $\frac{36}{792} = \frac{1}{22}$, so $m + n = 1 + 22 = 23$. Thus, **E** is the correct answer.

16. A point is chosen at random within the square in the coordinate plane whose vertices are $(0, 0)$, $(2020, 0)$, $(2020, 2020)$, and $(0, 2020)$. The probability that the point is within d units of a lattice point is $\frac{1}{2}$. (A point (x, y) is a lattice point if x and y are both integers.) What is d to the nearest tenth?

A 0.3

B 0.4

C 0.5

D 0.6

E 0.7

Solution:

For $d < \frac{1}{2}$, the points within d of lattice points occupy, in each unit square, four quarter-circles whose total area is πd^2 . The enormous square is tiled by unit squares, so the desired probability is πd^2 .

Setting $\pi d^2 = \frac{1}{2}$ gives $d = \sqrt{\frac{1}{2\pi}} \approx 0.399$, which rounds to 0.4. Thus, **B** is the correct answer.

17. Define

$$P(x) = (x - 1^2)(x - 2^2) \cdots (x - 100^2).$$

How many integers n are there such that $P(n) \leq 0$?

A 4900

B 4950

C 5000

D 5050

E 5100

Solution:

The polynomial changes sign at each square $1^2, 2^2, \dots, 100^2$, and its leading coefficient is positive. Thus $P(n) \leq 0$ for integers in the intervals $[1^2, 2^2], [3^2, 4^2], \dots, [99^2, 100^2]$.

For odd k , the interval $[k^2, (k+1)^2]$ contains $(k+1)^2 - k^2 + 1 = 2k + 2$ integers. Summing over odd $k = 1, 3, \dots, 99$ gives $2(1 + 3 + \dots + 99) + 2 \cdot 50 = 5000 + 100 = 5100$. Thus, **E** is the correct answer.

18. Let (a, b, c, d) be an ordered quadruple of not necessarily distinct integers, each one of them in the set $\{0, 1, 2, 3\}$. For how many such quadruples is it true that $a \cdot d - b \cdot c$ is odd? (For example, $(0, 3, 1, 1)$ is one such quadruple, because $0 \cdot 1 - 3 \cdot 1 = -3$ is odd.)

A 48

B 64

C 96

D 128

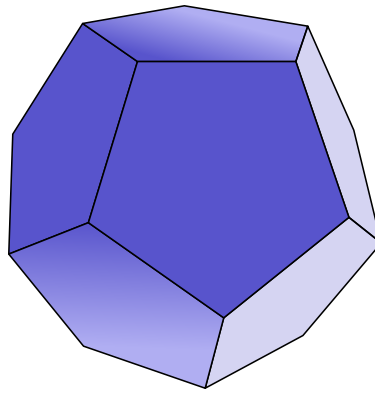
E 192

Solution:

Only parity matters. Modulo 2, the condition is that $ad - bc$ is 1, meaning the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible over $\mathbb{F}_2$.

There are $(4 - 1)(4 - 2) = 6$ invertible 2×2 matrices over $\mathbb{F}_2$. Each parity pattern lifts to $2^4 = 16$ choices from $\{0, 1, 2, 3\}$, so there are $6 \cdot 16 = 96$ quadruples. Thus, **C** is the correct answer.

19. As shown in the figure below, a regular dodecahedron (the polyhedron consisting of 12 congruent regular pentagonal faces) floats in space with two horizontal faces. Note that there is a ring of five slanted faces adjacent to the top face, and a ring of five slanted faces adjacent to the bottom face. How many ways are there to move from the top face to the bottom face via a sequence of adjacent faces so that each face is visited at most once and moves are not permitted from the bottom ring to the top ring?



- A 125
- B 250
- C 405
- D 640
- E 810**

Solution:

After leaving the top face, choose one of the 5 top-ring faces. Because moves from the bottom ring to the top ring are forbidden, every valid path has a top-ring phase, then one move down to the bottom ring, then a bottom-ring phase.

Fix the first top-ring face. On the top ring, the path can move around the 5-cycle without revisiting a face and then stop at any point: there are $1 + 2 \cdot 4 = 9$ possible top-ring paths. From the stopping face, there are 2 possible downward moves to the bottom ring, so the top part has 18 choices.

Once in the bottom ring, the path can move around the bottom 5-cycle without revisiting a face and then enter the bottom face; this gives $1 + 2 \cdot 4 = 9$ choices. The

total is $5 \cdot 18 \cdot 9 = 810$. Thus, **E** is the correct answer.

20. Quadrilateral $ABCD$ satisfies $\angle ABC = \angle ACD = 90^\circ$, $AC = 20$, and $CD = 30$. Diagonals $\overline{AC}$ and $\overline{BD}$ intersect at point E , and $AE = 5$. What is the area of quadrilateral $ABCD$?

A 330

B 340

C 350

D 360

E 370

Solution:

Place $A = (0, 0)$ and $C = (20, 0)$. Since $\angle ACD = 90^\circ$ and $CD = 30$, take $D = (20, 30)$. The point E is $(5, 0)$, so line BD has equation $y = 2(x - 5)$.

Because $\angle ABC = 90^\circ$, point B lies on the circle with diameter AC : $(x - 10)^2 + y^2 = 100$. Intersecting with $y = 2(x - 5)$ gives $x = 2$ or 10 . The convex quadrilateral uses $B = (2, -6)$.

Then $[ACD] = \frac{1}{2} \cdot 20 \cdot 30 = 300$, and $[ABC] = \frac{1}{2} \cdot 20 \cdot 6 = 60$. The total area is 360. Thus, **D** is the correct answer.

21. There exists a unique strictly increasing sequence of nonnegative integers $a_1 < a_2 < \dots < a_k$ such that

$$\frac{2^{289} + 1}{2^{17} + 1} = 2^{a_1} + 2^{a_2} + \dots + 2^{a_k}.$$

What is k ?

- ☐ A 117
- ☐ B 136
- ☒ C 137
- ☐ D 273
- ☐ E 306

Solution:

Let $X = 2^{17}$. Then

$$\begin{aligned} \frac{2^{289} + 1}{2^{17} + 1} &= \frac{X^{17} + 1}{X + 1} \\ &= X^{16} - X^{15} + X^{14} \\ &\quad - \dots - X + 1 \end{aligned}$$

Pair consecutive terms: $X^{16} - X^{15}, X^{14} - X^{13}, \dots, X^2 - X$, and then the final $+1$. Each pair is $2^{17m}(2^{17} - 1)$, contributing 17 ones in binary. There are 8 such pairs plus the final 1, so $k = 8 \cdot 17 + 1 = 137$. Thus, **C** is the correct answer.

22. For how many positive integers $n \leq 1000$ is

$$\left\lfloor \frac{998}{n} \right\rfloor + \left\lfloor \frac{999}{n} \right\rfloor + \left\lfloor \frac{1000}{n} \right\rfloor$$

not divisible by 3? (Recall that $\lfloor x \rfloor$ is the greatest integer less than or equal to x .)

A 22

B 23

C 24

D 25

E 26

Solution:

Write $1000 = qn + r$, where $0 \leq r < n$. Then $\left\lfloor \frac{1000}{n} \right\rfloor = q$. The other two floors are usually also q , except that subtracting 1 or 2 from 1000 crosses a multiple of n when r is small.

For $n > 1$, the sum is not divisible by 3 exactly when $r = 0$ or $r = 1$. The case $r = 0$ gives divisors of 1000, excluding 1, for 15 values. The case $r = 1$ gives divisors of 999, excluding 1, for $(3 + 1)(1 + 1) - 1 = 7$ values. The total is 22. Thus, **A** is the correct answer.

23. Let T be the triangle in the coordinate plane with vertices $(0, 0)$, $(4, 0)$, and $(0, 3)$. Consider the following five isometries (rigid transformations) of the plane: rotations of 90° , 180° , and 270° counterclockwise around the origin, reflection across the x -axis, and reflection across the y -axis. How many of the 125 sequences of three of these transformations (not necessarily distinct) will return T to its original position? (For example, a 180° rotation, followed by a reflection across the x -axis, followed by a reflection across the y -axis will return T to its original position, but a 90° rotation, followed by a reflection across the x -axis, followed by another reflection across the x -axis will not return T to its original position.)

A 12

B 15

C 17

D 20

E 25

Solution:

Let R be a 90° rotation, so the allowed rotations are R, R^2, R^3 . Let X and Y be the reflections across the coordinate axes. Once the first two transformations are chosen, the third is forced to be the inverse of their product.

Among two rotations, 6 ordered pairs have a nonidentity rotation as their product. A rotation and a reflection have an allowed axis-reflection as their product exactly when the rotation is R^2 , giving 4 ordered pairs. Finally, the two different axis-reflections can occur in either order, giving 2 more pairs. Altogether there are $6 + 4 + 2 = 12$ valid sequences. Thus, **A** is the correct answer.

24. Let n be the least positive integer greater than 1000 for which

$$\gcd(63, n + 120) = 21$$

and

$$\gcd(n + 63, 120) = 60.$$

What is the sum of the digits of n ?

A 12

B 15

C 18

D 21

E 24

Solution:

The first gcd condition gives $n + 120 \equiv 0 \pmod{21}$, so $n \equiv 6 \pmod{21}$, but $n + 120$ must not be divisible by 63. The second gives $n + 63 \equiv 0 \pmod{60}$, so $n \equiv 57 \pmod{60}$, but $n + 63$ must not be divisible by 120.

Solving $n \equiv 6 \pmod{21}$ and $n \equiv 57 \pmod{60}$ gives $n \equiv 237 \pmod{420}$. The candidates above 1000 are 1077, 1497, 1917, ... The first fails the first gcd condition, the second fails the second gcd condition, and 1917 works. The digit sum is 18. Thus, **C** is the correct answer.

25. Jason rolls three fair standard six-sided dice. Then he looks at the rolls and chooses a subset of the dice (possibly empty, possibly all three dice) to reroll. After rerolling, he wins if and only if the sum of the numbers face up on the three dice is exactly 7. Jason always plays to optimize his chances of winning. What is the probability that he chooses to reroll exactly two of the dice?

A $\frac{7}{36}$

B $\frac{5}{24}$

C $\frac{2}{9}$

D $\frac{17}{72}$

E $\frac{1}{4}$

Solution:

For any initial roll, Jason compares the best probabilities from rerolling 0, 1, 2, or 3 dice. Rerolling all three dice has probability $\frac{15}{216} = \frac{5}{72}$. Rerolling one die has probability $\frac{1}{6}$ whenever some pair of kept dice has sum at most 6.

If he rerolls exactly two dice, he keeps one die. Keeping a die showing 1, 2, 3, 4, 5, 6 gives probabilities 5, 4, 3, 2, 1, 0 out of 36, respectively. This can be optimal only when the two smallest dice sum at least 7 and the smallest die is 1, 2, or 3.

The sorted rolls satisfying this are (1, 6, 6), (2, 5, 5), (2, 5, 6), (2, 6, 6), and (3, 4, 4), (3, 4, 5), (3, 4, 6), (3, 5, 5), (3, 5, 6), (3, 6, 6). Counting permutations gives $3 + 12 + 27 = 42$ rolls out of 216, so the probability is $\frac{42}{216} = \frac{7}{36}$. Thus, **A** is the correct answer.

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