2019 AMC 10A Solutions

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1. What is the value of

$$2^{\left(0^{\left(1^9\right)}\right)}+\left(\left(2^0\right)^1\right)^9?$$

A 0

B 1

C 2

D 3

E 4

Solution:

We can evaluate from the top of each exponent tower downward:

$$\begin{aligned}&2^{\left(0^{\left(1^9\right)}\right)}\\&\quad+\left(\left(2^0\right)^1\right)^9\\&=2^{0^1}+\left(1^1\right)^9\\&=2^0+1^9\\&=2.\end{aligned}$$

Thus, **C** is the correct answer.

2. What is the hundreds digit of $(20! - 15!)$?

A 0

B 1

C 2

D 4

E 5

Solution:

Both $20!$ and $15!$ contain at least three factors of 5 and three factors of 2, so both are divisible by $2^3 \cdot 5^3 = 1000$.

Their difference is therefore also divisible by 1000.

Being a multiple of 1000 makes the last three digits 0, which shows that the hundreds digit is also 0.

Thus, **A** is the correct answer.

3. Ana and Bonita were born on the same date in different years, n years apart. Last year Ana was 5 times as old as Bonita. This year Ana's age is the square of Bonita's age. What is n?

A 3

B 5

C 9

D 12

E 15

Solution:

Let a be Ana's current age and b be Bonita's current age. Then

$$\begin{aligned} a-1 &= 5(b-1), \\ a &= b^2. \end{aligned}$$

Substitution gives

$$\begin{aligned} b^2-1 &= 5b-5, \\ b^2-5b+4 &= 0, \\ (b-4)(b-1) &= 0. \end{aligned}$$

We can see that $b \neq 1$ since that would make Ana and Bonita the same age, so we know that $b = 4$.

This gives us that $a = 4^2 = 16$ and $n = 16 - 4 = 12$.

Thus, **D** is the correct answer.

4. A box contains 28 red balls, 20 green balls, 19 yellow balls, 13 blue balls, 11 white balls, and 9 black balls. What is the minimum number of balls that must be drawn from the box without replacement to guarantee that at least 15 balls of a single color will be drawn?

A 75

B 76

C 79

D 84

E 91

Solution:

Note that we can pull as many as 14 balls of each color without ensuring that 15 balls of one color are drawn.

This means that we can draw all of the black, white and blue balls, along with 14 red, green, and yellow balls.

This gives us a total of

$$9 + 11 + 13 + 3 \cdot 14$$

$$= 33 + 42$$

$$= 75.$$

We need to add one at the end, however, to ensure that we get that 15th ball of some color, $75 + 1 = 76.$

Thus, **B** is the correct answer.

5. What is the greatest number of consecutive integers whose sum is 45?

A 9

B 25

C 45

D 90

E 120

Solution:

Suppose there are k consecutive integers with first term a. Their sum is $\frac{k(2a+k-1)}{2} = 45$, so k must divide 90. Therefore the number of terms cannot exceed 90.

This bound is attained by

$$-44, -43, \cdots, 44, 45$$

which has 90 terms and sum 45.

Thus the greatest possible number of consecutive integers is 90.

Thus, **D** is the correct answer.

6. For how many of the following types of quadrilaterals does there exist a point in the plane of the quadrilateral that is equidistant from all four vertices of the quadrilateral?

- a square
- a rectangle that is not a square
- a rhombus that is not a square
- a parallelogram that is not a rectangle or a rhombus
- an isosceles trapezoid that is not a parallelogram

A 1

B 2

C 3

D 4

E 5

Solution:

Note that if a point is equidistant from all the vertices, then that point is the center of the shape's circumcircle.

The question then becomes which of these shapes is cyclic (has a circumcircle). One condition that we can use is that opposite angles are supplementary.

Clearly, a square and rectangle that is not a square work (opposite angles are right, adding up to 180°).

A rhombus that is not a square does not work, since opposite angles are equal, but they are not 90°.

A parallelogram that is not a rectangle or a rhombus faces the same problem as above, making it not cyclic as well.

An isosceles trapezoid that is not a parallelogram by definition has supplementary opposite angles, making it cyclic.

Thus, **C** is the correct answer.

7. Two lines with slopes $\frac{1}{2}$ and 2 intersect at $(2, 2)$. What is the area of the triangle enclosed by these two lines and the line $x + y = 10$?

A 4

B $4\sqrt{2}$

C 6

D 8

E $6\sqrt{2}$

Solution:

The two lines through $(2, 2)$ have equations

$$y = \frac{1}{2}x + 1,$$
$$y = 2x - 2.$$

Their intersections with $x + y = 10$ are $(6, 4)$ and $(4, 6)$, respectively.

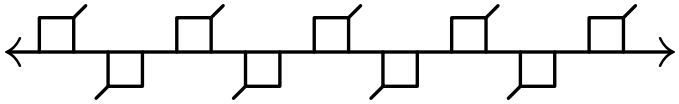
Thus the vertices are $(2, 2)$, $(6, 4)$, and $(4, 6)$. The segment joining the last two points has length $2\sqrt{2}$, its midpoint is $(5, 5)$, and the distance from $(2, 2)$ to that midpoint is $3\sqrt{2}$.

The area is therefore

$$\frac{1}{2}(2\sqrt{2})(3\sqrt{2}) = 6.$$

Thus, **C** is the correct answer.

8. The figure below shows line ℓ with a regular, infinite, recurring pattern of squares and line segments.



How many of the following four kinds of rigid motion transformations of the plane in which this figure is drawn, other than the identity transformation, will transform this figure into itself?

- some rotation around a point of line ℓ
- some translation in the direction parallel to line ℓ
- the reflection across line ℓ
- some reflection across a line perpendicular to line ℓ

A 0

B 1

C 2

D 3

E 4

Solution:

The first transformation works, as we can rotate ℓ 180° around the midpoint between an upward-facing and downward-facing square.

The second also works, as we can just move ℓ to the right until the squares line up with each other again.

The third fails, as a reflection would cause the line segments to face the opposite direction.

The fourth transformation also doesn't work since the diagonal lines would again be facing in the wrong direction.

Thus, **C** is the correct answer.

9. What is the greatest three-digit positive integer n for which the sum of the first n positive integers is *not* a divisor of the product of the first n positive integers?

A 995

B 996

C 997

D 998

E 999

Solution:

The sum of the first n numbers is

$$\frac{n(n+1)}{2}.$$

We need this to not divide $n!$.

Put $m = n + 1$. If m is composite, write $m = ab$ with $2 \leq a \leq b$. When $a < b$, the distinct factors a and b both occur in $(m-2)! = (n-1)!$. When $a = b$, we have $a \geq 3$, and the two multiples a and $2a$ both occur in $(m-2)!$, so $a^2 = m$ divides that factorial as well. Thus $m \mid (n-1)!$, and consequently

$$\frac{n(n+1)}{2} \mid n!.$$

Conversely, if $n + 1$ is prime, that prime factor does not occur in $n!$, so the divisibility fails. Since 997 is prime while $998, 999,$ and 1000 are composite, the greatest three-digit value is

$$997 - 1 = 996.$$

Thus, **B** is the correct answer.

10. A rectangular floor that is 10 feet wide and 17 feet long is tiled with 170 one-foot square tiles. A bug walks from one corner to the opposite corner in a straight line. Including the first and the last tile, how many tiles does the bug visit?

A 17

B 25

C 26

D 27

E 28

Solution:

Note that every time the bug crosses a vertical or horizontal line, the bug visits one new tile.

This means that the number of tiles the bug visits is 1 (the first tile) plus the number of lines it crosses.

The bug never walks over a corner since 10 and 17 are relatively prime, so we don't have to worry about that.

The bug crosses 16 horizontal lines and 9 vertical lines for a total of

$$1 + 16 + 9 = 26$$

tiles.

Thus, **C** is the correct answer.

11. How many positive integer divisors of 201^9 are perfect squares or perfect cubes (or both)?

A 32

B 36

C 37

D 39

E 41

Solution:

Taking the prime factorization of 201^9, we get $3^9 \cdot 67^9$.

Note that a perfect square has even exponents for its prime factors, and a cube's exponents are divisible by 3.

There are 5 options for an even exponent, from 0 through 8, and 4 options for multiples of 3, from 0 through 9.

This gives us 5^2 options for the squares and 4^2 options for the cubes. We have to subtract out the sixth powers, however.

Using the same logic, sixth powers have to have exponents of prime factors be divisible by 6. There are 2 options, 0 and 6.

This means that there are $2^2 = 4$ sixth powers. This gives us a total of

$$25 + 16 - 4 = 37.$$

perfect squares or perfect cubes.

Thus, **C** is the correct answer.

12. Melanie computes the mean μ, the median M, and the modes of the 365 values that are the dates in the months of 2019. Thus her data consist of 12 copies of 1, 12 copies of $2, \ldots, 12$ copies of 28, 11 copies of 29, 11 copies of 30, and 7 copies of 31. Let d be the median of the modes. Which of the following statements is true?

A $\mu < d < M$

B $M < d < \mu$

C $d = M = \mu$

D $d < M < \mu$

E $d < \mu < M$

Solution:

The modes are all the integers from 1 through 28, so their median is $d = \dfrac{14+15}{2} = 14.5$.

There are 365 entries, so M is the 183rd number. The dates from 1 through 15 occupy $15 \cdot 12 = 180$ positions, so $M = 16$.

The sum of all dates is

$$12(1 + \cdots + 28) + 11(29 + 30) + 7 \cdot 31 = 5738.$$

Hence $\mu = \dfrac{5738}{365} \approx 15.72$.

Therefore

$$d < \mu < M.$$

Thus, **E** is the correct answer.

13. Let $\triangle ABC$ be an isosceles triangle with $BC = AC$ and $\angle ACB = 40^\circ$. Construct the circle with diameter $\overline{BC}$, and let D and E be the other intersection points of the circle with the sides $\overline{AC}$ and $\overline{AB}$, respectively. Let F be the intersection of the diagonals of the quadrilateral $BCDE$. What is the degree measure of $\angle BFC$?

A 90

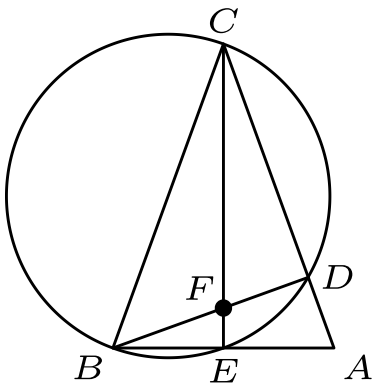
B 100

C 105

D 110

E 120

Solution:



Since $\overline{BC}$ is the diameter of the circle, we get that $\angle BDC$ and $\angle BEC$ are right angles.

We know that $\angle ABC = 70^\circ$ from the fact that $\triangle ABC$ is isosceles.

In $\triangle BCE$ and $\triangle BCD$, respectively,

$$\begin{aligned}\angle ECB &= 180^\circ - 70^\circ - 90^\circ \\ &= 20^\circ\end{aligned}$$

and

$$\begin{aligned}\angle DBC &= 180^\circ - 40^\circ - 90^\circ \\ &= 50^\circ.\end{aligned}$$

Because F lies on BD and CE, the other two angles of $\triangle BFC$ are 50° and 20°. Hence

$$\begin{aligned}\angle BFC &= 180^\circ - 50^\circ - 20^\circ \\ &= 110^\circ.\end{aligned}$$

Thus, **D** is the correct answer.

14. For a set of four distinct lines in a plane, there are exactly N distinct points that lie on two or more of the lines. What is the sum of all possible values of N?

A 14

B 16

C 18

D 19

E 21

Solution:

The values $0, 1, 3, 4, 5,$ and 6 are attainable. Four parallel lines give 0, four concurrent lines give 1, three parallel lines cut by a fourth give 3, three concurrent lines plus a fourth not through that point give 4, three lines forming a triangle plus a fourth parallel to one side give 5, and four lines in general position give $\binom{4}{2} = 6$.

It remains to rule out 2. Choose two nonparallel lines, meeting at X. If a third line also passes through X, then a fourth line not through X intersects at least two of those three concurrent lines at two different new points; otherwise all four lines pass through X, giving only one point. If the third line does not pass through X, then to create only one new point it must be parallel to one of the first two lines. A fourth distinct line cannot pass through either existing intersection without meeting the parallel line at a new point, and if it passes through neither, it creates a new intersection immediately. Thus exactly two intersection points are impossible.

Thus the possible values are $0, 1, 3, 4, 5, 6$, whose sum is 19. Thus, **D** is the correct answer.

15. A sequence of numbers is defined recursively by $a_1 = 1, a_2 = \frac{3}{7}$, and

$$a_n = \frac{a_{n-2} \cdot a_{n-1}}{2a_{n-2} - a_{n-1}}$$

for all $n \geq 3$. Then a_{2019} can be written as $\frac{p}{q}$, where p and q are relatively prime positive integers. What is $p + q$?

A 2020

B 4039

C 6057

D 6061

E 8078

Solution:

Taking reciprocals in the recursive formula gives

$$\begin{aligned} \frac{1}{a_n} &= \frac{2a_{n-2} - a_{n-1}}{a_{n-2}a_{n-1}} \\ &= \frac{2}{a_{n-1}} - \frac{1}{a_{n-2}}. \end{aligned}$$

This means that

$$\frac{1}{a_n} - \frac{1}{a_{n-1}} = \frac{1}{a_{n-1}} - \frac{1}{a_{n-2}},$$

which tells us that $\left\{ \frac{1}{a_n} \right\}$ is an arithmetic sequence.

Using a_1 and a_2, we get that the common difference is

$$\frac{1}{\frac{3}{7}} - \frac{1}{1} = \frac{7}{3} - 1 = \frac{4}{3}.$$

Therefore

$$\frac{1}{a_{2019}} = 1 + 2018 \cdot \frac{4}{3} = \frac{8075}{3},$$

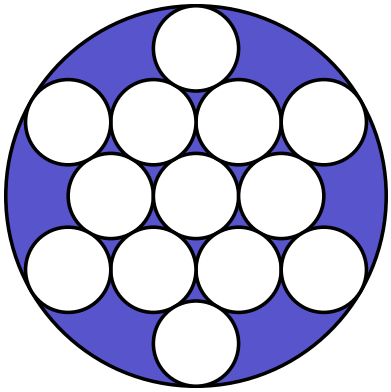
so $a_{2019} = \frac{3}{8075}$.

$p + q$ is therefore

$$8075 + 3 = 8078.$$

Thus, **E** is the correct answer.

16. The figure below shows 13 circles of radius 1 within a larger circle. All the intersections occur at points of tangency. What is the area of the region, shaded in the figure, inside the larger circle but outside all the circles of radius 1?



A $4\pi\sqrt{3}$

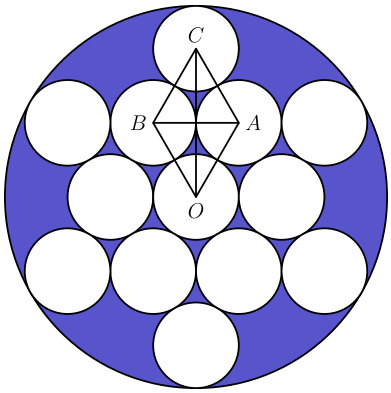
B 7π

C $\pi\left(3\sqrt{3}+2\right)$

D $10\pi\left(\sqrt{3}-1\right)$

E $\pi\left(\sqrt{3}+6\right)$

Solution:



We know $\triangle ABC$ and $\triangle ABO$ are equilateral triangles.

We get that $OC = 2\sqrt{3}$ using special right triangles to find the altitudes of the triangles.

The radius of the larger circle is therefore $2\sqrt{3}+1$, since there is the extra unit radius after $\overline{OC}$.

The area of the larger circle is

$$(2\sqrt{3}+1)^2\pi = (13+4\sqrt{3})\pi.$$

The area of all the inner circles is 13π.

The area of the shaded region is

$$(13+4\sqrt{3})\pi - 13\pi = 4\pi\sqrt{3}.$$

Thus, **A** is the correct answer.

17. A child builds towers using identically shaped cubes of different colors. How many different towers with a height 8 cubes can the child build with 2 red cubes, 3 blue cubes, and 4 green cubes? (One cube will be left out.)

A 24

B 288

C 312

D $1{,}260$

E $40{,}320$

Solution:

Given a valid height-8 tower, its color counts determine the one unused cube; place that cube on top. Conversely, removing the top cube from any arrangement of all 9 cubes gives a valid height-8 tower. These operations are inverses, so the desired towers are in one-to-one correspondence with arrangements of all 9 cubes.

There are $9!$ ways to make a tower of height 9, but we are overcounting since there are multiple cubes of the same color.

We have to divide through by $2!$ ways to arrange the red cubes, $3!$ for the blue cubes, and $4!$ for the green cubes.

Therefore, the number of valid arrangements is

$$\frac{9!}{2! \cdot 3! \cdot 4!} = 1,260.$$

Thus, **D** is the correct answer.

18. For some positive integer k, the repeating base-k representation of the (base-ten) fraction $\frac{7}{51}$ is

$$0.\overline{23}_k = 0.232323\ldots_k.$$

What is k?

A 13

B 14

C 15

D 16

E 17

Solution:

The repeating base-k fraction is $2k^{-1} + 3k^{-2} + 2k^{-3} + 3k^{-4} + \cdots$. Grouping odd and even powers gives

$$\begin{aligned} &2(k^{-1} + k^{-3} + \cdots) \\ &\quad + 3(k^{-2} + k^{-4} + \cdots). \end{aligned}$$

Using geometric series, these sums are $\frac{2k}{k^2-1}$ and $\frac{3}{k^2-1}$, so $0.\overline{23}_k = \frac{2k+3}{k^2-1}$.

Setting $\frac{2k+3}{k^2-1} = \frac{7}{51}$ gives $51(2k+3) = 7(k^2-1)$, so $7k^2 - 102k - 160 = 0$.

Hence $k = 16$. Thus, **D** is the correct answer.

19. What is the least possible value of

$$(x+1)(x+2)(x+3)(x+4) + 2019,$$

where x is a real number?

A 2017

B 2018

C 2019

D 2020

E 2021

Solution:

Multiplying the first two terms and the last terms yields

$$(x^2+5x+4)(x^2+5x+6).$$

Note that these two terms differ by 2. We can try to express this as a difference of squares, which is

$$(x^2+5x+5)^2-1.$$

Adding 2019 to this gets us

$$(x^2+5x+5)^2+2018.$$

Squares are non-negative, so as long as we find a way to make the inner expression 0, we can make the square 0.

The discriminant is $5^2-4\cdot 5=5$, which is positive meaning that there is a value that makes the square 0.

This means that the minimum value would be

$$0^2+2018=2018.$$

Thus, **B** is the correct answer.

20. The numbers $1, 2, \ldots, 9$ are randomly placed into the 9 squares of a 3×3 grid. Each square gets one number, and each of the numbers is used once. What is the probability that the sum of the numbers in each row and each column is odd?

A $\frac{1}{21}$

B $\frac{1}{14}$

C $\frac{5}{63}$

D $\frac{2}{21}$

E $\frac{1}{7}$

Solution:

Note that the only way to get an odd sum is if there are either 0 or 2 even numbers in the row or column.

The only way for this to happen is if the 4 even numbers form a rectangle with sides parallel to the large square.

The way to see this is we choose a spot for the first even number. Then we need to choose another square in the same row, x, and column, y, to be even.

The final even has to be in same column as x and the same row as y. This forms the aforementioned rectangle.

There are four 2×2 rectangles, two 3×2 rectangles, two 2×3 rectangles, and one 3×3 rectangle.

This gives a total of

$$4 + 2 + 2 + 1 = 9$$

possible sets of positions for the even numbers.

There are $4!$ ways to arrange the even numbers and $5!$ ways to arrange the odd numbers.

This means that there are a total of

$$9 \cdot 4! \cdot 5!$$

configurations of squares that satisfy the condition.

There are a total of $9!$ arrangements with no restrictions. The probability is therefore

$$\frac{9 \cdot 4! \cdot 5!}{9!} = \frac{1}{14}.$$

Thus, **B** is the correct answer.

21. A sphere with center O has radius 6. A triangle with sides of length $15, 15$, and 24 is situated in space so that each of its sides is tangent to the sphere. What is the distance between O and the plane determined by the triangle?

A $2\sqrt{3}$

B 4

C $3\sqrt{2}$

D $2\sqrt{5}$

E 5

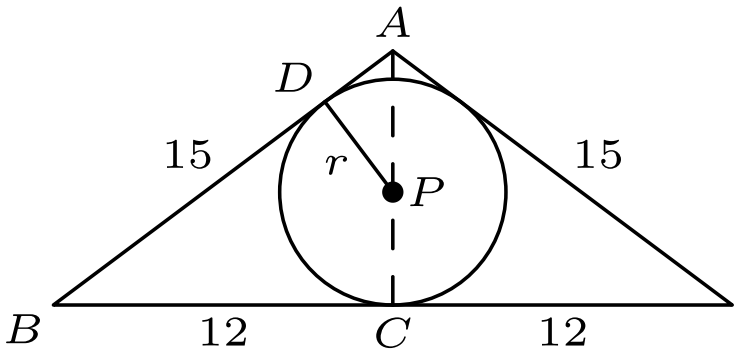
Solution:

The altitude to the side of length 24 is

$$\sqrt{15^2 - 12^2} = 9,$$

so the triangle has area $\frac{1}{2} \cdot 24 \cdot 9 = 108$ and semiperimeter 27. Its inradius is therefore

$$r = \frac{108}{27} = 4.$$



Let P be the perpendicular projection of O onto the triangle's plane, and let $d = OP$. Because all three side-lines are tangent to the sphere, P is the incenter and its perpendicular distance to each side is 4. The distance from O to each side-line is the sphere's radius, 6, so the Pythagorean theorem gives

$$d^2 + 4^2 = 6^2.$$

Hence $d = 2\sqrt{5}$.

Thus, **D** is the correct answer.

22. Real numbers between 0 and 1, inclusive, are chosen in the following manner. A fair coin is flipped. If it lands heads, then it is flipped again and the chosen number is 0 if the second flip is heads, and 1 if the second flip is tails. On the other hand, if the first coin flip is tails, then the number is chosen uniformly at random from the closed interval $[0, 1]$. Two random numbers x and y are chosen independently in this manner. What is the probability that $|x - y| > \frac{1}{2}$?

A $\frac{1}{3}$

B $\frac{7}{16}$

C $\frac{1}{2}$

D $\frac{9}{16}$

E $\frac{2}{3}$

Solution:

We can case on whether x and y are chosen from the interval or from 0 and 1. Each case has a $\frac{1}{4}$ chance of happening, since they depend on two coin flips.

Case 1: x and y are either 0 or 1

x and y need to be different, which happens with a $\frac{1}{2}$ probability.

Case 2: x is either 0 or 1, and y is chosen from $[0, 1]$

If $x = 0$, then y has to be chosen from $\left(\frac{1}{2}, 1\right]$, and if $x = 1$, then y has to be chosen from $\left[0, \frac{1}{2}\right)$.

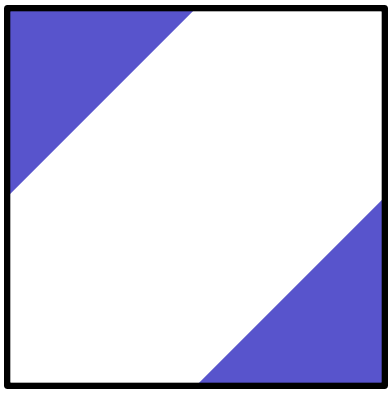
This means that y always has a $\frac{1}{2}$ probability of being chosen from the correct interval.

Case 3: x is chosen from $[0, 1]$, and y is either 0 or 1

This has the same probability as case 2 due to symmetry.

Case 4: x and y are chosen from $[0, 1]$

We can use geometric probability since we are working with an infinite number of (x, y) pairs. We graph $|x - y| > \frac{1}{2}$.



The shaded area covers $\frac{1}{4}$ of the graph, showing that there is a $\frac{1}{4}$ probability of this case working.

Adding the four weighted probabilities gives

$$\frac{1}{4}\left(\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{4}\right) = \frac{7}{16}.$$

Thus, **B** is the correct answer.

23. Travis has to babysit the terrible Thompson triplets. Knowing that they love big numbers, Travis devises a counting game for them. First Tadd will say the number 1, then Todd must say the next two numbers (2 and 3), then Tucker must say the next three numbers ($4, 5, 6$), then Tadd must say the next four numbers ($7, 8, 9, 10$), and the process continues to rotate through the three children in order, each saying one more number than the previous child did, until the number $10{,}000$ is reached. What is the 2019th number said by Tadd?

A 5743

B 5885

C 5979

D 6001

E 6011

Solution:

Tadd speaks on turns of lengths $1, 4, 7, \ldots$. After n of Tadd's turns, he has said

$$\sum_{i=1}^{n}(3i-2) = \frac{3n^2-n}{2}$$

numbers.

For $n = 36$, this total is 1926, while for $n = 37$ it is 2035. Therefore Tadd's 2019th number is the $(2019 - 1926) = 93$rd number of his 37th turn.

Before that turn, the children have completed turns of lengths 1 through 108, saying $1 + 2 + \cdots + 108 = 5886$ numbers. The 93rd number of the next turn is $5886 + 93 = 5979$. Thus, **C** is the correct answer.

24. Let p, q, and r be the distinct roots of the polynomial

$$x^3 - 22x^2 + 80x - 67.$$

There exist real numbers A, B, and C such that

$$\begin{aligned}\frac{1}{s^3 - 22s^2 + 80s - 67} &= \frac{A}{s-p} + \frac{B}{s-q} \\ &\quad + \frac{C}{s-r}\end{aligned}$$

for all real numbers s with $s \notin \{p, q, r\}$. What is

$$\frac{1}{A} + \frac{1}{B} + \frac{1}{C}?$$

A 243

B 244

C 245

D 246

E 247

Solution:

Multiplying the identity by $(s-p)(s-q)(s-r)$ gives

$$\begin{aligned}1 &= A(s-q)(s-r) \\ &\quad + B(s-p)(s-r) \\ &\quad + C(s-p)(s-q).\end{aligned}$$

Setting $s = p, q, r$, in turn, yields

$$\begin{aligned}\frac{1}{A} &= (p-q)(p-r),\\ \frac{1}{B} &= (q-p)(q-r),\\ \frac{1}{C} &= (r-p)(r-q).\end{aligned}$$

Adding and expanding gives

$$\begin{aligned}\frac{1}{A}+\frac{1}{B}+\frac{1}{C} &= p^2+q^2+r^2\\ &\quad -pq-pr-qr.\end{aligned}$$

By Vieta's formulas, $p+q+r = 22$ and $pq+pr+qr = 80$, so

$$\begin{aligned}p^2+q^2+r^2 &= 22^2-2(80)\\ &= 324.\end{aligned}$$

Therefore the requested value is $324-80 = 244.$

Thus, **B** is the correct answer.

25. For how many integers n between 1 and 50, inclusive, is

$$\frac{(n^2-1)!}{(n!)^n}$$

an integer? (Recall that $0! = 1$.)

A 31

B 32

C 33

D 34

E 35

Solution:

One fact that greatly helps with this problem is realizing that

$$\frac{(n^2)!}{(n!)^{n+1}}$$

is always an integer.

This is because it is the number of ways to split up n^2 objects into n unordered groups of size n.

Now, we get that

$$\frac{(n^2-1)!}{(n!)^n} = \frac{(n^2)!}{(n!)^{n+1}} \cdot \frac{n!}{n^2}.$$

Therefore, whenever n^2 divides $n!$, the original expression is an integer; this is equivalent to n dividing $(n-1)!$.

Suppose n is composite. If $n = ab$ with $2 \leq a < b < n$, then the distinct factors a and b both occur in $(n-1)!$, so $n \mid (n-1)!$. If $n = a^2$ with $a \geq 3$, then $(n-1)!$ contains the distinct factors a and $2a$, whose product is a multiple of n. Thus every composite $n \neq 4$ works. The case $n = 1$ also works directly.

Conversely, if $n = p$ is prime, the exponent of p in the denominator is p, while its exponent in $(p^2 - 1)!$ is $p - 1$, so the expression is not an integer.

For $n = 4$, the denominator contains 2^{12}, while $15!$ contains only 2^{11}, so this case also fails.

There are 15 primes at most 50, and adding 4, we get 16 values for n that do not work.

Therefore, the desired answer is $50 - 16 = 34$.

Thus, **D** is the correct answer.

Problems: https://live.poshenloh.com/past-contests/amc10/2019A