

2018 AMC 10A Solutions

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1. What is the value of

$$\left(\left((2 + 1)^{-1} + 1 \right)^{-1} + 1 \right)^{-1} + 1?$$

- A $\frac{5}{8}$
- B $\frac{11}{7}$**
- C $\frac{8}{5}$
- D $\frac{18}{11}$
- E $\frac{15}{8}$

Solution:

We can simplify this as follows.

$$\begin{aligned} & \left(\left((2 + 1)^{-1} + 1 \right)^{-1} + 1 \right)^{-1} + 1 \\ &= \left(\left(\left(\frac{1}{3} + 1 \right)^{-1} + 1 \right)^{-1} + 1 \right) \\ &= \left(\frac{3}{4} + 1 \right)^{-1} + 1 \\ &= \frac{4}{7} + 1 \\ &= \frac{11}{7} \end{aligned}$$

Thus, **B** is the correct answer.

2. Liliane has 50% more soda than Jacqueline, and Alice has 25% more soda than Jacqueline. What is the relationship between the amounts of soda that Liliane and Alice have?

- ☒ A Liliane has 20% more soda than Alice.
- ☐ B Liliane has 25% more soda than Alice.
- ☐ C Liliane has 45% more soda than Alice.
- ☐ D Liliane has 75% more soda than Alice.
- ☐ E Liliane has 100% more soda than Alice.

Solution:

Let x be the number of gallons of soda that Jacqueline has. Then Alice has $1.25x$ gallons, and Liliane has $1.5x$ gallons.

Therefore, the relationship can be found by dividing the amount of soda that each has to yield $\frac{1.5x}{1.25x} = 1.2$, which means Liliane has 20% more soda than Alice.

Thus, **A** is the correct answer.

3. A unit of blood expires after

$$10! = 10 \cdot 9 \cdot 8 \cdots 1$$

seconds. Yasin donates a unit of blood at noon on January 1. On what day does his unit of blood expire?

- ☐ A January 2
- ☐ B January 12
- ☐ C January 22
- ☐ D February 11
- ☒ E February 12

Solution:

We can divide $10!$ by 60 , 60 , and 24 to get the number of days that it takes for a unit of blood to expire.

The first division cancels a 6 and 10 . The second division cancels 3 , 4 , and 5 . The final division cancels 8 and turns the 9 into a 3 .

This leaves 2 , 7 , and a 3 , which multiply to 42 . There are 31 days in January, so by February 1, the blood only has $42 - 31 = 11$ days left.

11 days from February 1 would make the blood expire on February 12.

Thus, **E** is the correct answer.

4. How many ways can a student schedule 3 mathematics courses — algebra, geometry, and number theory — in a 6-period day if no two mathematics courses can be taken in consecutive periods?

(What courses the student takes during the other 3 periods is of no concern here.)

- A 3
- B 6
- C 12
- D 18
- E 24**

Solution:

The 3 classes can occupy the following periods:

(1, 3, 5),

(1, 3, 6),

(1, 4, 6),

(2, 4, 6).

This means that there are 4 ways to choose which periods the mathematics courses occur.

For each configuration, there are $3!$ ways to determine the order of the courses, for a total of $6 \cdot 4 = 24$ schedules.

Thus, **E** is the correct answer.

5. Alice, Bob, and Charlie were on a hike and were wondering how far away the nearest town was. When Alice said, “We are at least 6 miles away,” Bob replied, “We are at most 5 miles away.” Charlie then remarked, “Actually the nearest town is at most 4 miles away.” It turned out that none of the three statements was true. Let d be the distance in miles to the nearest town. Which of the following intervals is the set of all possible values of d ?

A $(0, 4)$

B $(4, 5)$

C $(4, 6)$

D $(5, 6)$

E $(5, \infty)$

Solution:

Alice’s statement tells us that $d < 6$. Bob’s statement tells us that $d > 5$. Charlie’s statement tells us that $d > 4$.

Combining all of these tells us that $5 < d$ and $d < 6$, which means d is in the interval $(5, 6)$.

Thus, **D** is the correct answer.

6. Sangho uploaded a video to a website where viewers can vote that they like or dislike a video. Each video begins with a score of 0, and the score increases by 1 for each like vote and decreases by 1 for each dislike vote.

At one point Sangho saw that his video had a score of 90, and that 65% of the votes cast on his video were like votes. How many votes had been cast on Sangho's video at that point?

- ☐ A 200
- ☒ B 300
- ☐ C 400
- ☐ D 500
- ☐ E 600

Solution:

If 65% of votes were like votes, then 35% of votes are dislike votes. Then Sangho's score is $65\% - 35\% = 30\%$ the total number of votes.

We know that Sangho's score is 90, so the total number of votes is $90 \div 30\% = 300$.

Thus, **B** is the correct answer.

7. For how many (not necessarily positive) integer values of n is the following value an integer?

$$4000 \cdot \left(\frac{2}{5}\right)^n$$

- A 3
- B 4
- C 6
- D 8
- E 9**

Solution:

We can rewrite the expression as

$$(2^5 \cdot 5^3) \cdot \left(\frac{2}{5}\right)^n = 2^{5+n} \cdot 5^{3-n}.$$

For this to be an integer, both exponents must be nonnegative. This means that

$$5 + n \geq 0 \Rightarrow n \geq -5$$

$$3 - n \geq 0 \Rightarrow n \leq 3.$$

This gives us $5 + 3 + 1 = 9$ values for n .

Thus, **E** is the correct answer.

8. Joe has a collection of 23 coins, consisting of 5-cent coins, 10-cent coins, and 25-cent coins. He has 3 more 10-cent coins than 5-cent coins, and the total value of his collection is 320 cents. How many more 25-cent coins does Joe have than 5-cent coins?

- ☐ A 0
- ☐ B 1
- ☒ C 2
- ☐ D 3
- ☐ E 4

Solution:

Let x be the number of 5-cent coins that Joe has. Then the number of 10-cent coins he has is $x + 3$.

Therefore, Joe has

$$23 - x - (x + 3) = 20 - 2x$$

25-cent coins.

The total value of all these coins is

$$\begin{aligned} 5x + 10(x + 3) + 25(20 - 2x) \\ = 530 - 35x. \end{aligned}$$

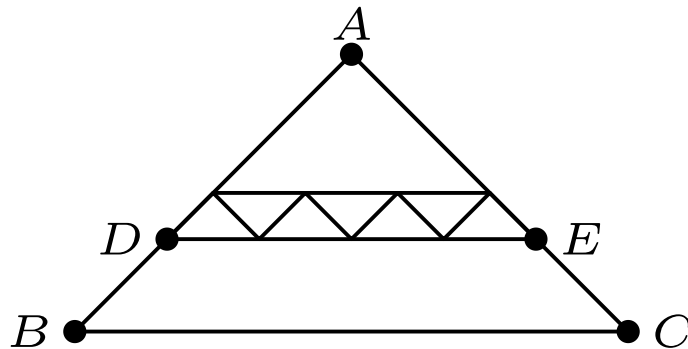
We know that

$$530 - 35x = 320 \Rightarrow x = 6.$$

This means that Joe has $20 - 2 \cdot 6 = 8$ 25-cent coins. Therefore, he has $8 - 6 = 2$ more 25-cent coins than 5-cent coins.

Thus, **C** is the correct answer.

9. All of the triangles in the diagram below are similar to isosceles triangle ABC , in which $AB = AC$. Each of the 7 smallest triangles has area 1, and $\triangle ABC$ has area 40. What is the area of trapezoid $DBCE$?



- ☐ A 16
- ☐ B 18
- ☐ C 20
- ☐ D 22
- ☒ E 24

Solution:

We know that the side length of the smaller triangles is $\sqrt{\frac{1}{40}}$ times the length of the larger triangle from similar triangles.

Then the side length of $\triangle ADE$ is $4\sqrt{\frac{1}{40}}$ times the length of the side length of the larger triangle.

This makes the ratio of the areas

$$\left(4\sqrt{\frac{1}{40}}\right)^2 = 16 \cdot \frac{1}{40} = \frac{2}{5}.$$

Therefore, the area of $\triangle ADE$ is $\frac{2}{5} \cdot 40 = 16$. The area of the trapezoid is then $40 - 16 = 24$.

Thus, **E** is the correct answer.

10. Suppose that real number x satisfies

$$\sqrt{49 - x^2} - \sqrt{25 - x^2} = 3.$$

What is the value of

$$\sqrt{49 - x^2} + \sqrt{25 - x^2}?$$

A 8

B $\sqrt{33} + 8$

C 9

D $2\sqrt{10} + 4$

E 12

Solution:

Note that the left hand side of the equation and the desired expression are conjugates. Multiplying them would remove the square roots.

Multiplying them yields

$$49 - x^2 - 25 + x^2 = 24.$$

This means that the product of the values of the expressions is equal to 24. The desired value is therefore $24 \div 3 = 8$.

Thus, **A** is the correct answer.

11. When 7 fair standard 6-sided dice are thrown, the probability that the sum of the numbers on the top faces is 10 can be written as

$$\frac{n}{6^7},$$

where n is a positive integer. What is n ?

A 42

B 49

C 56

D 63

E 84

Solution:

We can use stars and bars to find n . It is the same as finding the number of ways to put 10 balls into 7 boxes, where each box has at least one ball.

The formula for such a scenario is

$$\binom{n-1}{k-1},$$

where n is the number of balls and k is the number of boxes.

No die can exceed 4 in a sum of 10 from seven positive rolls, so the upper bound of 6 creates no additional restriction. The desired answer is therefore

$$\binom{9}{6} = \binom{9}{3} = 84.$$

Thus, **E** is the correct answer.

12. How many ordered pairs of real numbers (x, y) satisfy the following system of equations?

$$\begin{cases} x + 3y &= 3 \\ ||x| - |y|| &= 1 \end{cases}$$

- ☐ A 1
- ☐ B 2
- ☒ C 3
- ☐ D 4
- ☐ E 8

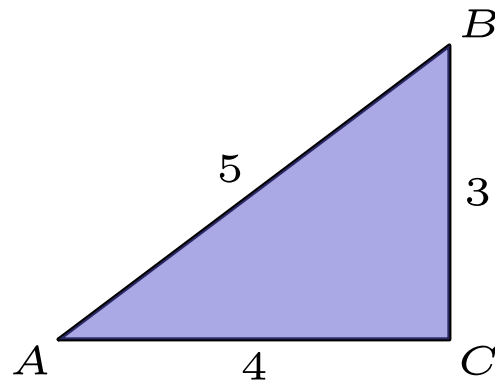
Solution:

The second equation says $|x| - |y| = 1$ or $|x| - |y| = -1$, so it is enough to check the four linear possibilities $x = y \pm 1$ and $x = -y \pm 1$.

Combining these with $x + 3y = 3$ gives $(x, y) = \left(\frac{3}{2}, \frac{1}{2}\right), (0, 1), (0, 1)$ again, and $(-3, 2)$.

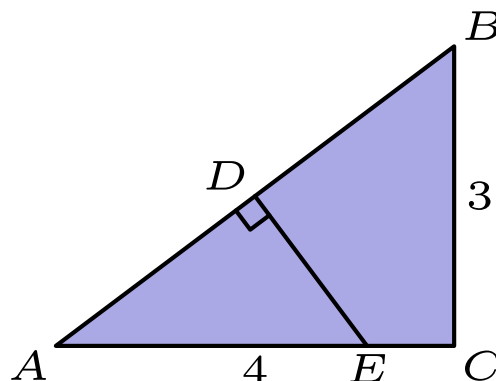
These are three distinct ordered pairs, and each satisfies the original absolute-value equation. Thus, **C** is the correct answer.

13. A paper triangle with sides of lengths 3, 4, and 5 inches, as shown, is folded so that point A falls on point B . What is the length in inches of the crease?



- A $1 + \frac{1}{2}\sqrt{2}$
- B $\sqrt{3}$
- C $\frac{7}{4}$
- D $\frac{15}{8}$**
- E 2

Solution:



The crease is the perpendicular bisector of $\overline{AB}$. Let $\overline{DE}$ be the crease.

By AA similarity, $\triangle ADE \sim \triangle ACB$. Therefore,

$$\frac{BC}{AC} = \frac{DE}{AD}.$$

Plugging in the side lengths gives

$$\frac{3}{4} = \frac{DE}{\frac{5}{2}},$$

$$\text{so } DE = \frac{15}{8}.$$

Thus, **D** is the correct answer.

14. What is the greatest integer less than or equal to

$$\frac{3^{100} + 2^{100}}{3^{96} + 2^{96}}?$$

A 80

B 81

C 96

D 97

E 625

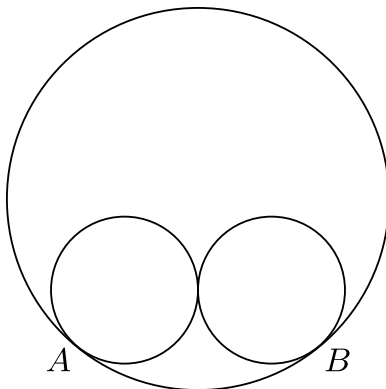
Solution:

Let $a = 3^{96}$ and $b = 2^{96}$. The expression is $\frac{81a + 16b}{a + b} = 16 + \frac{65a}{a + b}$, so it is less than $16 + 65 = 81$.

To show the floor is 80, we also need the expression to be greater than 80. This is equivalent to $81a + 16b > 80a + 80b$, or $a > 64b$.

Because $\left(\frac{3}{2}\right)^2 = \frac{9}{4} > 2$, we have $\frac{a}{b} = \left(\frac{3}{2}\right)^{96} > 2^{48} > 64$. Hence the expression is greater than 80 and less than 81. Thus, **A** is the correct answer.

15. Two circles of radius 5 are externally tangent to each other and are internally tangent to a circle of radius 13 at points A and B , as shown in the diagram. The distance AB can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n$?



- ☐ A 21
- ☐ B 29
- ☐ C 58
- ☒ D 69
- ☐ E 93

Solution:

Let X be the center of the large circle and let Y, Z be the centers of the two smaller circles. Then $XY = XZ = 13 - 5 = 8$ and $YZ = 10$.

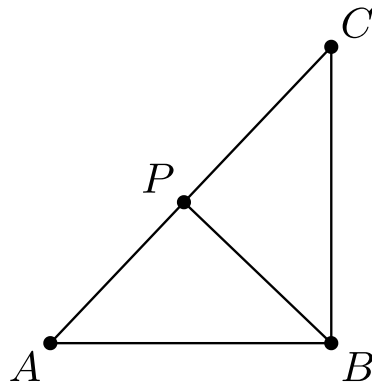
The radii to tangent points make XA collinear with XY and XB collinear with XZ , so $\triangle XAB \sim \triangle XYZ$. Thus $\frac{AB}{YZ} = \frac{XA}{XY} = \frac{13}{8}$.

Hence $AB = 10 \cdot \frac{13}{8} = \frac{65}{4}$, so $m + n = 65 + 4 = 69$. Thus, **D** is the correct answer.

16. Right triangle ABC has leg lengths $AB = 20$ and $BC = 21$. Including $\overline{AB}$ and $\overline{BC}$, how many line segments with integer length can be drawn from vertex B to a point on hypotenuse $\overline{AC}$?

- A 5
- B 8
- C 12
- D 13**
- E 15

Solution:



Let P be the foot of the altitude from B to $\overline{AC}$. The Pythagorean Theorem gives $AC = 29$. Computing the area in two ways gives

$$\frac{29 \cdot PB}{2} = \frac{20 \cdot 21}{2},$$

so $PB = \frac{420}{29}$, which lies between 14 and 15.

As the endpoint moves from A to P , its distance from B decreases continuously from 20 to PB . Thus there is one segment of each integer length 15, 16, 17, 18, 19, 20. As the endpoint moves from P to C , the distance increases continuously from PB to 21, giving one segment of each integer length 15, 16, 17, 18, 19, 20, 21. These are $6 + 7 = 13$ distinct segments.

Thus, **D** is the correct answer.

17. Let S be a set of 6 integers taken from $\{1, 2, \dots, 12\}$ with the property that if a and b are elements of S with $a < b$, then b is not a multiple of a . What is the least possible value of an element in S ?

- A 2
- B 3
- C 4**
- D 5
- E 7

Solution:

We proceed by casing on possible values for S :

1 cannot be the smallest element since that would mean that no other number can be in the set.

2 cannot be the smallest element since we would have to include every odd number except 1. This would make 3 and 9 violate the rule.

Let 3 be the smallest element. Then we can include 7 and 11. We can finally include either 4 or 8 and 5 or 10.

Either way, the maximum number of elements that we can include is 5, so 3 cannot be the smallest element.

Starting with 4, we can include 6, 7, 9 and 11. Finally, we can add either 5 or 10, creating a 6-element set.

Thus, **C** is the correct answer.

18. How many nonnegative integers can be written in the form

$$\begin{aligned} & a_7 \cdot 3^7 + a_6 \cdot 3^6 + a_5 \cdot 3^5 \\ & + a_4 \cdot 3^4 + a_3 \cdot 3^3 + a_2 \cdot 3^2 \\ & + a_1 \cdot 3^1 + a_0 \cdot 3^0, \end{aligned}$$

where $a_i \in \{-1, 0, 1\}$ for $0 \leq i \leq 7$?

- A 512
- B 729
- C 1094
- D 3281**
- E 59,048

Solution:

Note that every number formed by this sum is either positive, negative, or zero.

The number of positive numbers equals the number of negative numbers due to symmetry (flip the 1 s to -1 s and -1 s to 1 s).

The only way for the sum to be 0 is if all the coefficients are 0.

The total number of numbers is $3^8 = 6561$. Because each power of 3 is larger than the sum of all previous powers of three, each combination of coefficients yields a different value. More explicitly, at the highest place where two combinations differ, the difference has magnitude at least 3^k , while all lower places together can cancel at most $2(1 + 3 + \cdots + 3^{k-1}) = 3^k - 1$.

Therefore, there are

$$\frac{6561 - 1}{2} + 1 = 3281$$

distinct nonnegative integers.

Thus, **D** is the correct answer.

19. A number m is randomly selected from the set

$$\{11, 13, 15, 17, 19\},$$

and a number n is randomly selected from

$$\{1999, 2000, 2001, \dots, 2018\}.$$

What is the probability that m^n has a units digit of 1?

A $\frac{1}{5}$

B $\frac{1}{4}$

C $\frac{3}{10}$

D $\frac{7}{20}$

E $\frac{2}{5}$

Solution:

Only the units digit of m matters. Among the 20 consecutive possible exponents, each residue modulo 4 occurs 5 times, and 10 exponents are even. A base ending in 1 succeeds for all 20 exponents; bases ending in 3 or 7 succeed for the 5 exponents divisible by 4; a base ending in 5 never succeeds; and a base ending in 9 succeeds for the 10 even exponents.

Thus $20 + 5 + 0 + 5 + 10 = 40$ of the $5 \cdot 20 = 100$ equally likely pairs work, and the probability is $\frac{40}{100} = \frac{2}{5}$.

Thus, **E** is the correct answer.

20. A scanning code consists of a 7×7 grid of squares, with some of its squares colored black and the rest colored white. There must be at least one square of each color in this grid of 49 squares.

A scanning code is called *symmetric* if its look does not change when the entire square is rotated by a multiple of 90° counterclockwise around its center, nor when it is reflected across a line joining opposite corners or a line joining midpoints of opposite sides.

What is the total number of possible symmetric scanning codes?

- A 510
- B 1022**
- C 8190
- D 8192
- E 65,534

Solution:

Number rows and columns from -3 through 3 , with the center at $(0, 0)$. Rotations and reflections can change signs and exchange coordinates, so the orbit of (x, y) is determined by the ordered pair obtained by sorting $(|x|, |y|)$. These pairs are (u, v) with $0 \leq u \leq v \leq 3$, of which there are $4 + 3 + 2 + 1 = 10$.

Once one square in each orbit is colored, symmetry forces the colors of all other squares in that orbit.

There are therefore 2^{10} symmetric colorings before the condition about using both colors. The all-black and all-white colorings are not allowed.

The total number of valid symmetric scanning codes is $2^{10} - 2 = 1022$. Thus, **B** is the correct answer.

21. Which of the following describes the set of values of a for which the curves $x^2 + y^2 = a^2$ and $y = x^2 - a$ in the real xy -plane intersect at exactly 3 points?

A $a = \frac{1}{4}$

B $\frac{1}{4} < a < \frac{1}{2}$

C $a > \frac{1}{4}$

D $a = \frac{1}{2}$

E $a > \frac{1}{2}$

Solution:

Substitute $y = x^2 - a$ into $x^2 + y^2 = a^2$. This gives $x^2 + (x^2 - a)^2 = a^2$, so $x^2(x^2 - (2a - 1)) = 0$.

The factor $x^2 = 0$ always gives the single point $(0, -a)$. The other factor gives two additional real points exactly when $2a - 1 > 0$.

There are exactly three intersection points when $a > \frac{1}{2}$. Thus, **E** is the correct answer.

22. Let a, b, c , and d be positive integers such that

$$\gcd(a, b) = 24,$$

$$\gcd(b, c) = 36,$$

$$\gcd(c, d) = 54,$$

and

$$70 < \gcd(d, a) < 100.$$

Which of the following must be a divisor of a ?

A 5

B 7

C 11

D 13

E 17

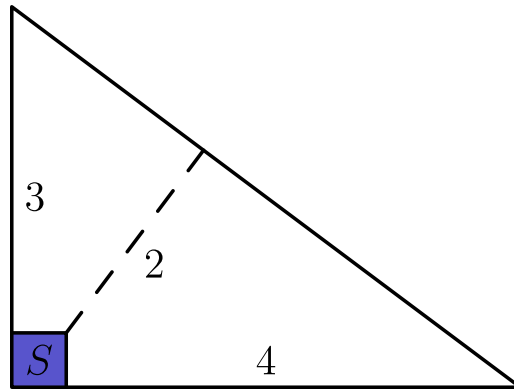
Solution:

From $\gcd(a, b) = 24 = 2^3 \cdot 3$ and $\gcd(b, c) = 36 = 2^2 \cdot 3^2$, the number a is divisible by $2^3 \cdot 3$ but not by 3^2 .

From $\gcd(b, c) = 36$ and $\gcd(c, d) = 54 = 2 \cdot 3^3$, the number d is divisible by $2 \cdot 3^3$ but not by 2^2 . Therefore $\gcd(d, a) = 2 \cdot 3 \cdot n$, where n has no factor 2 or 3.

Since $70 < 6n < 100$, the integer n satisfies $12 \leq n \leq 16$. The only value in this range with no factor 2 or 3 is 13, so 13 must divide a . Thus, **D** is the correct answer.

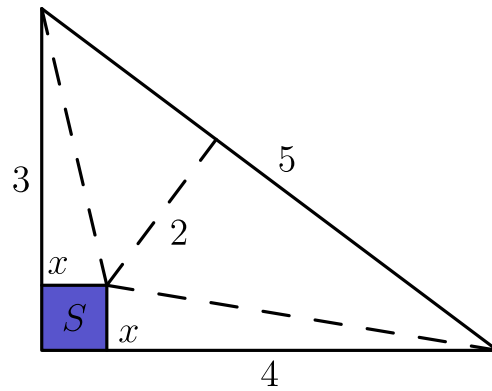
23. Farmer Pythagoras has a field in the shape of a right triangle. The right triangle's legs have lengths 3 and 4 units. In the corner where those sides meet at a right angle, he leaves a small unplanted square S so that from the air it looks like the right angle symbol. The rest of the field is planted. The shortest distance from S to the hypotenuse is 2 units. What fraction of the field is planted?



- A $\frac{25}{27}$
- B $\frac{26}{27}$
- C $\frac{73}{75}$
- D $\frac{145}{147}$**
- E $\frac{74}{75}$

Solution:

Let x be the side length of S . Then we can split the field up into the following shapes.



We can express the area of the field in two ways:

$$\frac{3 \cdot 4}{2} = x^2 + \frac{x(3-x)}{2} + \frac{x(4-x)}{2} + \frac{2 \cdot 5}{2}.$$

Simplifying yields

$$6 = \frac{7x}{2} + 5$$

$$x = \frac{2}{7}.$$

The desired fraction is

$$\frac{6 - x^2}{6} = \frac{6 - \frac{4}{49}}{6} = \frac{145}{147}.$$

Thus, **D** is the correct answer.

24. Triangle ABC with $AB = 50$ and $AC = 10$ has area 120. Let D be the midpoint of $\overline{AB}$, and let E be the midpoint of $\overline{AC}$. The angle bisector of $\angle BAC$ intersects $\overline{DE}$ and $\overline{BC}$ at F and G , respectively. What is the area of quadrilateral $FDBG$?

A 60

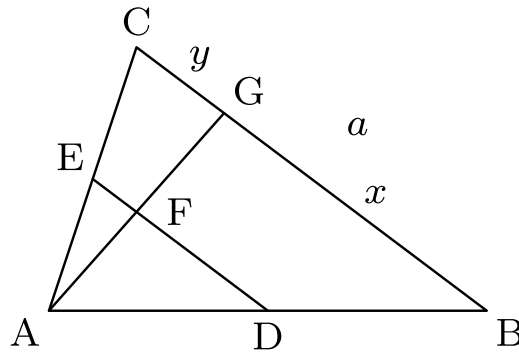
B 65

C 70

D 75

E 80

Solution:



Let $BC = a$, $BG = x$, $GC = y$, and h be the length of the altitude through A .

By the Angle Bisector Theorem, $BG : GC = AB : AC = 5 : 1$, so $BG = \frac{5a}{6}$.

Because DE is a midsegment, $DE = \frac{a}{2}$. Applying the same theorem in $\triangle ADE$

gives $DF : FE = AD : AE = 5 : 1$, so $DF = \frac{5}{6}DE = \frac{5a}{12}$.

The trapezoid's height is $\frac{h}{2}$, and $\frac{ah}{2} = 120$. Its average base length is

$\frac{1}{2} \left(\frac{5a}{12} + \frac{5a}{6} \right) = \frac{5a}{8}$. Therefore, its area is

$$\frac{5a}{8} \cdot \frac{h}{2} = \frac{5}{8} \cdot \frac{ah}{2} = 75.$$

Thus, **D** is the correct answer.

25. For a positive integer n and nonzero digits a , b , and c , let A_n be the n -digit integer each of whose digits is equal to a ; let B_n be the n -digit integer each of whose digits is equal to b ; and let C_n be the $2n$ -digit (not n -digit) integer each of whose digits is equal to c . What is the greatest possible value of $a + b + c$ for which there are at least two values of n such that

$$C_n - B_n = A_n^2?$$

- ☐ A 12
- ☐ B 14
- ☐ C 16
- ☒ D 18
- ☐ E 20

Solution:

The repeated-digit numbers can be written as

$$\begin{aligned} A_n &= a \frac{10^n - 1}{9}, \\ B_n &= b \frac{10^n - 1}{9}, \\ C_n &= c \frac{10^{2n} - 1}{9}. \end{aligned}$$

Substituting these expressions into $C_n - B_n = A_n^2$ and dividing by $\frac{10^n - 1}{9}$ gives

$$c(10^n + 1) - b = a^2 \frac{10^n - 1}{9}.$$

Rearranging yields

$$(9c - a^2)10^n = 9b - 9c - a^2.$$

If this holds for two different values of n , subtracting the two equations shows that $9c - a^2 = 0$. The displayed equation then also forces $9b - 9c - a^2 = 0$.

Hence $c = \frac{a^2}{9}$ and $b = \frac{2a^2}{9}$. Because a, b, c are nonzero digits, the candidates are $(a, b, c) = (3, 2, 1), (6, 8, 4)$, and $(9, 18, 9)$. The last triple is invalid because 18 is not a digit. The greatest valid sum is $6 + 8 + 4 = 18$. Thus, **D** is the correct answer.

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