

2017 AMC 10B Solutions

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1. Mary thought of a positive two-digit number. She multiplied it by 3 and added 11. Then she switched the digits of the result, obtaining a number between 71 and 75, inclusive. What was Mary's number?

- ☐ A 11
- ☒ B 12
- ☐ C 13
- ☐ D 14
- ☐ E 15

Solution:

Work backward from the possible results. Reversing 71, 72, 73, 74, 75 gives 17, 27, 37, 47, 57, respectively. Subtracting 11 gives 6, 16, 26, 36, 46. Only 6 and 36 are divisible by 3, and only $36 \div 3 = 12$ is a two-digit number.

Thus, the correct answer is **B**.

2. Sofia ran 5 laps around the 400-meter track at her school. For each lap, she ran the first 100 meters at an average speed of 4 meters per second and the remaining 300 meters at an average speed of 5 meters per second. How much time did Sofia take running the 5 laps?

A 5 minutes and 35 seconds

B 6 minutes and 40 seconds

C 7 minutes and 5 seconds

D 7 minutes and 25 seconds

E 8 minutes and 10 seconds

Solution:

She ran a total of $5 \cdot 100 = 500$ meters at 4 meters per second and $5 \cdot 300 = 1500$ meters at 5 meters per second.

Therefore, her time is

$$\frac{500}{4} + \frac{1500}{5} = 425$$

seconds.

This is equal to a total of 7 minutes and 5 seconds.

Thus, the correct answer is **C**.

3. Real numbers x , y , and z satisfy the inequalities $0 < x < 1$, $-1 < y < 0$, and $1 < z < 2$.

Which of the following numbers is necessarily positive?

- A $y + x^2$
- B $y + xz$
- C $y + y^2$
- D $y + 2y^2$
- E $y + z$

Solution:

Since $-1 < y$ and $1 < z$, we can add the inequalities to see that $0 < y + z$. This naturally proves choice **E** correct.

Furthermore, we can eliminate every other choice with the following values:

$$x = 0.1,$$

$$y = -0.25,$$

$$z = 1.25.$$

Thus, the correct answer is **E**.

4. Suppose that x and y are nonzero real numbers such that

$$\frac{3x + y}{x - 3y} = -2.$$

What is the value of

$$\frac{x + 3y}{3x - y}?$$

- A -3
- B -1
- C 1
- D 2**
- E 3

Solution:

Given that

$$\frac{3x + y}{x - 3y} = -2,$$

we can multiply by the denominator to get

$$3x + y = 6y - 2x.$$

Solving, we can see that $x = y$.

Therefore,

$$\frac{x + 3y}{3x - y} = \frac{x + 3x}{3x - x} = 2.$$

Thus, the correct answer is **D**.

5. Camilla had twice as many blueberry jelly beans as cherry jelly beans. After eating 10 pieces of each kind, she now has three times as many blueberry jelly beans as cherry jelly beans. How many blueberry jelly beans did she originally have?

A 10

B 20

C 30

D 40

E 50

Solution:

Let the number of cherry jelly beans be c and let the number of blueberry jelly beans be b .

Then, we know

$$b = 2c$$

$$b - 10 = 3(c - 10)$$

from the first and second statements respectively.

Therefore,

$$2c - 10 = 3c - 30$$

$$c = 20.$$

This means that

$$b = 2 \cdot 20 = 40.$$

Thus, the correct answer is **D**.

6. What is the largest number of solid $2\text{-in} \times 2\text{-in} \times 1\text{-in}$ blocks that can fit in a $3\text{-in} \times 2\text{-in} \times 3\text{-in}$ box?

- ☐ A 3
- ☒ B 4
- ☐ C 5
- ☐ D 6
- ☐ E 7

Solution:

The volume of the large solid object is $3 \cdot 3 \cdot 2 = 18$ and volume of the smaller object is $2 \cdot 2 \cdot 1 = 4$. This means we can fit at most 4 of the small objects.

We can make this happen by putting 3 of the small objects in a $3 \times 2 \times 2$ rectangular prism, and then we have a $3 \times 2 \times 1$ space left where we can place one small object.

Thus, the correct answer is **B**.

7. Samia set off on her bicycle to visit her friend, traveling at an average speed of 17 kilometers per hour. When she had gone half the distance to her friend's house, a tire went flat, and she walked the rest of the way at 5 kilometers per hour.

In all, it took her 44 minutes to reach her friend's house. In kilometers rounded to the nearest tenth, how far did Samia walk?

A 2.0

B 2.2

C 2.8

D 3.4

E 4.4

Solution:

Let d be the distance Samia walked. She bicycled the same distance, so her total travel time gives

$$\frac{d}{17} + \frac{d}{5} = \frac{44}{60}.$$

Solving yields $d = \frac{17}{6} = 2.833 \dots$, which rounds to 2.8 kilometers.

Thus, the correct answer is **C**.

8. Points $A(11, 9)$ and $B(2, -3)$ are vertices of $\triangle ABC$ with $AB = AC$. The altitude from A meets the opposite side at $D(-1, 3)$. What are the coordinates of point C ?

A $(-8, 9)$

B $(-4, 8)$

C $(-4, 9)$

D $(-2, 3)$

E $(-1, 0)$

Solution:

Since $AB = AC$, the altitude from A also bisects the base BC . Therefore, D is the midpoint of BC . If $C = (x, y)$, then we have

$$\frac{x + 2}{2} = -1,$$
$$\frac{y - 3}{2} = 3.$$

As such,

$$C = (x, y) = (-4, 9).$$

Thus, the correct answer is **C**.

9. A radio program has a quiz consisting of 3 multiple-choice questions, each with 3 choices. A contestant wins if he or she gets 2 or more of the questions right. The contestant answers randomly to each question. What is the probability of winning?

A $\frac{1}{27}$

B $\frac{1}{9}$

C $\frac{2}{9}$

D $\frac{7}{27}$

E $\frac{1}{2}$

Solution:

The probability that a contestant gets all 3 correct is

$$\left(\frac{1}{3}\right)^3 = \frac{1}{27}.$$

The probability of getting exactly 2 correct is

$$\binom{3}{2} \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right) = \frac{6}{27}.$$

The combined probability is

$$\frac{6}{27} + \frac{1}{27} = \frac{7}{27}.$$

Thus, the correct answer is **D**.

10. The lines with equations $ax - 2y = c$ and $2x + by = -c$ are perpendicular and intersect at $(1, -5)$. What is c ?

A -13

B -8

C 2

D 8

E 13

Solution:

The first equation can be rewritten as $y = \frac{a}{2}x - \frac{c}{2}$, and the second as $y = -\frac{2}{b}x - \frac{c}{b}$.

Because the lines are perpendicular, their slopes multiply to -1 , so $a = b$.

Substituting $(x, y) = (1, -5)$ into the two original equations gives $a + 10 = c$ and $2 - 5a = -c$. Adding these equations yields $12 - 4a = 0$, so $a = 3$ and $c = 13$.

Thus, the correct answer is **E**.

11. At Typico High School, 60% of the students like dancing, and the rest dislike it. Of those who like dancing, 80% say that they like it, and the rest say that they dislike it. Of those who dislike dancing, 90% say that they dislike it, and the rest say that they like it. What fraction of students who say they dislike dancing actually like it?

A 10%

B 12%

C 20%

D 25%

E $33\frac{1}{3}\%$

Solution:

Observe that of the 60% of people that actually like dancing, only 80% say they like dancing. This suggests that 48% of the students say that they like dancing, and as such, $60\% - 48\% = 12\%$ of the students who like dancing say they don't like it.

Then, we know that 90% of the 40% of people who don't like dancing say they don't like it, which is 36% of the total student population.

This means the total amount of people who say they don't like dancing is $12\% + 36\% = 48\%$.

We know then that the fraction of people who say they dislike dancing but actually like it is equal to:

$$\frac{12}{48} = \frac{1}{4} = 25\%.$$

Thus, the correct answer is **D**.

12. Elmer's new car gets 50% better fuel efficiency, measured in kilometers per liter, than his old car. However, his new car uses diesel fuel, which is 20% more expensive per liter than the gasoline his old car uses. By what percent will Elmer save money if he uses his new car instead of his old car for a long trip?

A 20%

B $26\frac{2}{3}\%$

C $27\frac{7}{9}\%$

D $33\frac{1}{3}\%$

E $41\frac{2}{3}\%$

Solution:

Let the old car's fuel efficiency be M kilometers per liter and let gasoline cost C dollars per liter. The old car therefore costs $\frac{C}{M}$ dollars per kilometer to fuel.

The new car gets $1.5M$ kilometers per liter and its fuel costs $1.2C$ dollars per liter, so its fuel cost per kilometer is

$$\frac{1.2C}{1.5M} = 0.8 \frac{C}{M}.$$

The new fuel cost is 80% of the old one, so Elmer saves 20%.

Thus, the correct answer is **A**.

13. There are 20 students participating in an after-school program offering classes in yoga, bridge, and painting. Each student must take at least one of these three classes, but may take two or all three.

There are 10 students taking yoga, 13 taking bridge, and 9 taking painting. There are 9 students taking at least two classes. How many students are taking all three classes?

- ☐ A 1
- ☐ B 2
- ☒ C 3
- ☐ D 4
- ☐ E 5

Solution:

The number of classes taken total is $10 + 13 + 9 = 32$.

Let x represent the number of people who take 1, let y represent the number of people who take 2 classes, and let z represent the number of people who take 3 classes.

Then, we know $x + 2y + 3z = 32$.

As such, the total number of people is 20, so $x + y + z = 20$. This makes $y + 2z = 12$.

The number of people who take at least two classes is 9, so $y + z = 9$.

Therefore, $z = 3$, making that the answer.

Thus, the correct answer is **C**.

14. An integer N is selected at random in the range $1 \leq N \leq 2020$. What is the probability that the remainder when N^{16} is divided by 5 is 1?

A $\frac{1}{5}$

B $\frac{2}{5}$

C $\frac{3}{5}$

D $\frac{4}{5}$

E 1

Solution:

By Fermat's Little Theorem, $N^4 \equiv 1 \pmod{5}$ whenever N is not divisible by 5. Therefore,

$$N^{16} \equiv (N^4)^4 \equiv 1 \pmod{5}.$$

There are $\frac{2020}{5} = 404$ multiples of 5, so there are $2020 - 404 = 1616$ allowable values of N .

A multiple of 5 has $N^{16} \equiv 0 \pmod{5}$, so no other values work. Thus the probability is

$$\frac{1616}{2020} = \frac{4}{5}.$$

Thus, the correct answer is **D**.

15. Rectangle $ABCD$ has $AB = 3$ and $BC = 4$. Point E is the foot of the perpendicular from B to diagonal $\overline{AC}$. What is the area of $\triangle AED$?

A 1

B $\frac{42}{25}$

C $\frac{28}{15}$

D 2

E $\frac{54}{25}$

Solution:

The area of $\triangle CDA$ is $\frac{3 \cdot 4}{2} = 6$. Since E lies on AC , triangles EAD and CDA share the same altitude from D , so $[EAD] = [CDA] \cdot \frac{AE}{AC}$.

By the Pythagorean Theorem, $AC = 5$. Also $\triangle ABE \sim \triangle ACB$, so $\frac{AE}{AB} = \frac{AB}{AC} = \frac{3}{5}$, giving $AE = \frac{9}{5}$. Thus $\frac{AE}{AC} = \frac{9}{25}$.

Therefore $[EAD] = 6 \cdot \frac{9}{25} = \frac{54}{25}$. Thus, **E** is the correct answer.

16. How many of the base-ten numerals for the positive integers less than or equal to 2017 contain the digit 0?

A 469

B 471

C 475

D 478

E 481

Solution:

For numbers less than 100, we only have a 0 if it is a multiple of 10, of which there are 9.

For numbers between 100 and 999 inclusive, we will use complementary counting. There are 900 total numbers in this range. Also, there are $9 \cdot 9 \cdot 9 = 729$ numbers in this range with no 0 since there are 9 ways to choose each digit to not be 0. Thus, the total in this range is 171.

For numbers between 1000 and 1999 inclusive, we will use complementary counting again. There are 1000 total numbers in this range. Also, there are $1 \cdot 9 \cdot 9 \cdot 9 = 729$ numbers in this range with no 0 since there are 9 ways to choose each of the last 3 digits to not be 0 and the first digit must be 1. Thus, the total in this range is 271.

There are 18 numbers between 2000 and 2017 inclusive, each with a 0 in the second digit from the left.

This makes the total

$$9 + 171 + 271 + 18 = 469.$$

Thus, the correct answer is **A**.

17. Call a positive integer *monotonous* if it is a one-digit number or its digits, when read from left to right, form either a strictly increasing or a strictly decreasing sequence. For example, 3, 23578, and 987620 are monotonous, but 88, 7434, and 23557 are not. How many monotonous positive integers are there?

A 1024

B 1524

C 1533

D 1536

E 2048

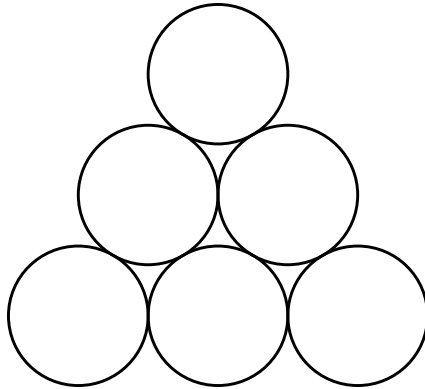
Solution:

The strictly increasing positive integers correspond to the nonempty subsets of $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, written in increasing order. There are $2^9 - 1 = 511$ of these.

The strictly decreasing positive integers correspond to subsets of $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, written in decreasing order, except for the empty set and $\{0\}$. There are $2^{10} - 2 = 1022$ of these.

The one-digit numbers 1 through 9 were counted in both groups, so the total is $511 + 1022 - 9 = 1524$. Thus, **B** is the correct answer.

18. In the figure below, 3 of the 6 disks are to be painted blue, 2 are to be painted red, and 1 is to be painted green. Two paintings that can be obtained from one another by a rotation or a reflection of the entire figure are considered the same. How many different paintings are possible?



- ☐ A 6
- ☐ B 8
- ☐ C 9
- ☒ D 12
- ☐ E 15

Solution:

By symmetry, the green disk has two possible types of position: a corner or a side midpoint. Fix one representative of either type. There are $\binom{5}{2} = 10$ ways to choose the two red disks.

The reflection that fixes the green position fixes one of the other disks and exchanges the other four disks in two pairs. Exactly 2 red-disk choices are unchanged by this reflection: choosing either exchanged pair. The other 8 choices form 4 mirror-image pairs. Hence there are $2 + 4 = 6$ paintings for each type of green position.

The two types therefore give $6 + 6 = 12$ paintings.

Thus, the correct answer is **D**.

19. Let ABC be an equilateral triangle. Extend side $\overline{AB}$ beyond B to a point B' so that $BB' = 3 \cdot AB$. Similarly, extend side $\overline{BC}$ beyond C to a point C' so that $CC' = 3 \cdot BC$, and extend side $\overline{CA}$ beyond A to a point A' so that $AA' = 3 \cdot CA$.

What is the ratio of the area of $\triangle A'B'C'$ to the area of $\triangle ABC$?

A 9 : 1

B 16 : 1

C 25 : 1

D 36 : 1

E 37 : 1

Solution:

Let X be the area of $\triangle ABC$. Each of $\triangle BB'C$, $\triangle CC'A$, and $\triangle AA'B$ has a base three times as long as a side of $\triangle ABC$ and the same corresponding altitude. Each therefore has area $3X$.

Next, $\triangle AA'C'$ has three times the base and the same altitude as $\triangle ACC'$, whose area is $3X$. Thus $\triangle AA'C'$ has area $9X$. Similarly, $\triangle CC'B'$ and $\triangle BB'A'$ each have area $9X$.

These seven regions partition the large triangle, so

$$\begin{aligned}[A'B'C'] &= X + 3(3X) + 3(9X) \\ &= 37X.\end{aligned}$$

The requested ratio is 37 : 1.

Thus, the correct answer is **E**.

20. The number $21! = 51,090,942,171,709,440,000$ has over 60,000 positive integer divisors. One of them is chosen at random. What is the probability that it is odd?

A $\frac{1}{21}$

B $\frac{1}{19}$

C $\frac{1}{18}$

D $\frac{1}{2}$

E $\frac{11}{21}$

Solution:

The exponent of 2 in $21!$ is

$$\begin{aligned} & \left\lfloor \frac{21}{2} \right\rfloor + \left\lfloor \frac{21}{4} \right\rfloor \\ & \quad + \left\lfloor \frac{21}{8} \right\rfloor + \left\lfloor \frac{21}{16} \right\rfloor \\ & = 10 + 5 + 2 + 1 = 18. \end{aligned}$$

Thus $21! = 2^{18}d$ for some odd integer d .

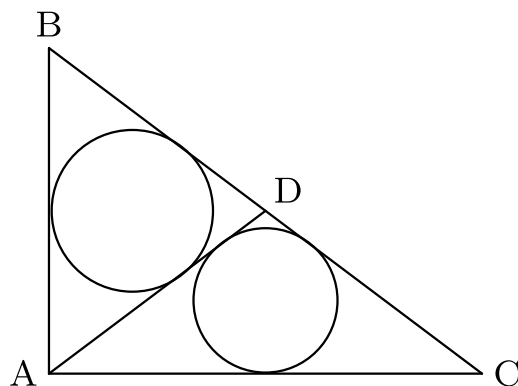
For every divisor x of d , the divisors of $21!$ with odd part x are $x, 2x, \dots, 2^{18}x$. Exactly one of these 19 divisors is odd, so the probability is $\frac{1}{19}$.

Thus, the correct answer is **B**.

21. In $\triangle ABC$, $AB = 6$, $AC = 8$, $BC = 10$, and D is the midpoint of $\overline{BC}$. What is the sum of the radii of the circles inscribed in $\triangle ADB$ and $\triangle ADC$?

- A $\sqrt{5}$
- B $\frac{11}{4}$
- C $2\sqrt{2}$
- D $\frac{17}{6}$**
- E 3

Solution:



The triangle ABC is a right triangle with a right angle at A . This makes D the circumcenter of the triangle since it is the midpoint of the hypotenuse.

Therefore,

$$AD = BD = DC = 5.$$

Also, the area of ABC is

$$\frac{6 \cdot 8}{2} = 24.$$

The bases BD and DC are equal, and the two triangles share the same altitude from A . Therefore, $\triangle ABD$ and $\triangle ACD$ each have area 12.

Then, for each triangle, we have $A = rs$ where A is the area, r is the inradius, and s is the semiperimeter. Equivalently, $12 = \frac{1}{2}rP$, where P is the perimeter, so $r = \frac{24}{P}$. For $\triangle ABD$, the inradius is $\frac{24}{5+5+6} = \frac{3}{2}$. For $\triangle ACD$, it is $\frac{24}{5+5+8} = \frac{4}{3}$.

Their sum is $\frac{3}{2} + \frac{4}{3} = \frac{17}{6}$.

Thus, the correct answer is **D**.

22. The diameter $\overline{AB}$ of a circle of radius 2 is extended to a point D outside the circle so that $BD = 3$. Point E is chosen so that $ED = 5$ and line ED is perpendicular to line AD . Segment $\overline{AE}$ intersects the circle at a point C between A and E . What is the area of $\triangle ABC$?

A $\frac{120}{37}$

B $\frac{140}{39}$

C $\frac{145}{39}$

D $\frac{140}{37}$

E $\frac{120}{31}$

Solution:

Since the radius is 2 and $BD = 3$, we have $AD = 7$. Since $ED = 5$ and the angle at D is a right angle, the area of ADE is $\frac{5 \cdot 7}{2} = \frac{35}{2}$.

By the Pythagorean Theorem, $AE = \sqrt{5^2 + 7^2} = \sqrt{74}$. Also, $\angle ACB$ is a right angle because AB is a diameter. The triangles share the angle at A , so $\triangle ABC \sim \triangle AED$ by angle-angle similarity.

Their corresponding hypotenuses are $AB = 4$ and $AE = \sqrt{74}$, so their area ratio is

$$\frac{[ABC]}{[AED]} = \left(\frac{4}{\sqrt{74}} \right)^2 = \frac{8}{37}.$$

Therefore,

$$[ABC] = \frac{8}{37} \cdot \frac{35}{2} = \frac{140}{37}.$$

Thus, the correct answer is **D**.

23. Let

$$N = 123456789101112 \dots 4344$$

be the 79-digit number that is formed by writing the integers from 1 to 44 in order, one after the other. What is the remainder when N is divided by 45?

- A 1
- B 4
- C 9**
- D 18
- E 44

Solution:

To find the remainder when divided by 45, we must find the remainder when divided by 5 and 9. The remainder when divided by 5 is the remainder when the units digit is divided by 5, making it 4.

To find the remainder when divided by 9, we usually find the sum of the digits. However, each double digit number has the same remainder when divided by 9 as its digit sum, so we can just sum the integers from 1 to 44, because each integer is congruent to its own digit sum. This sum is $\frac{44 \cdot 45}{2} = 990$, which is a multiple of 9.

Thus, N is a multiple of 9.

Since it is a multiple of 9 and has a remainder of 4 when divided by 5, the remainder when divided by 45 is 9.

Thus, the correct answer is **C**.

24. The vertices of an equilateral triangle lie on the hyperbola $xy = 1$, and a vertex of this hyperbola is the centroid of the triangle. What is the square of the area of the triangle?

A 48

B 60

C 108

D 120

E 169

Solution:

By symmetry, assume that the centroid is the hyperbola vertex $G = (1, 1)$. At least two triangle vertices lie on the same branch of the hyperbola. They cannot both lie on the negative branch: if two of their x -coordinates were negative, the third would exceed 3, while the sum of the three y -coordinates would be less than $\frac{1}{3}$, contradicting that their centroid is $(1, 1)$. Thus two vertices lie on the positive branch.

Write these vertices as $P = (a, \frac{1}{a})$ and $Q = (b, \frac{1}{b})$, where $a, b > 0$. The centroid of an equilateral triangle is also its circumcenter, so P and Q are equidistant from G . For $t > 0$, the squared distance from $(t, \frac{1}{t})$ to G is

$$\left(t + \frac{1}{t}\right) \left(t + \frac{1}{t} - 2\right).$$

This is strictly increasing as $t + \frac{1}{t}$ increases from 2, so the distinct points must satisfy $b = \frac{1}{a}$. Hence P and Q are reflections across $y = x$.

The third vertex lies on the perpendicular bisector $y = x$. Its coordinates also satisfy $xy = 1$, so it is either $(1, 1)$ or $(-1, -1)$. It cannot equal the centroid, so it is $(-1, -1)$. Therefore, the circumradius is the distance from $(1, 1)$ to $(-1, -1)$, namely $2\sqrt{2}$.

Dividing the equilateral triangle into three triangles at its center gives its area as

$$3 \cdot \frac{1}{2} (2\sqrt{2})^2 \sin 120^\circ = 6\sqrt{3}.$$

The square of the area is $(6\sqrt{3})^2 = 108$.

Thus, the correct answer is **C**.

25. Last year Isabella took 7 math tests and received 7 different scores, each an integer between 91 and 100, inclusive. After each test she noticed that the average of her test scores was an integer. Her score on the seventh test was 95. What was her score on the sixth test?

A 92

B 94

C 96

D 98

E 100

Solution:

Let S be the sum of all seven scores. Since all seven averages were integers, S is divisible by 7. Also the seven distinct scores are between 91 and 100, so $91 + 92 + \cdots + 97 \leq S \leq 94 + 95 + \cdots + 100$.

Thus $658 \leq S \leq 679$, and the possible multiples of 7 are 658, 665, 672, 679. Since the seventh score is 95, the first six scores sum to $S - 95$, which must be divisible by 6. This forces $S = 665$.

The first six scores sum to 570. The first five-score average was also an integer, so the sum of the first five scores is divisible by 5. Therefore the sixth score is divisible by 5. Since the seventh score is already 95 and all scores are distinct, the sixth score is 100. Thus, **E** is the correct answer.

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