

2016 AMC 10B Solutions

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1. What is the value of

$$\frac{2a^{-1} + \frac{a^{-1}}{2}}{a}$$

when $a = \frac{1}{2}$?

- ☐ A 1
- ☐ B 2
- ☐ C $\frac{5}{2}$
- ☒ D 10
- ☐ E 20

Solution:

The expression is equivalent to

$$2a^{-2} + \frac{a^{-2}}{2} = 2.5(a^{-1})^2.$$

Then a^{-1} is equal to $\frac{1}{\frac{1}{2}} = 2$, so our expression is equal to $2.5 \cdot 2^2 = 10$.

Thus, the correct answer is **D**.

2. If $n \heartsuit m = n^3 m^2$, what is $\frac{2 \heartsuit 4}{4 \heartsuit 2}$?

A $\frac{1}{4}$

B $\frac{1}{2}$

C 1

D 2

E 4

Solution:

For any nonzero a, b , we have

$$\frac{a \heartsuit b}{b \heartsuit a} = \frac{a^3 b^2}{b^3 a^2} = \frac{a}{b}.$$

Using $a = 2, b = 4$, we get that $\frac{2}{4} = \frac{1}{2}$.

Thus, the correct answer is **B**.

3. Let $x = -2016$. What is the value of

$$\left| \left| |x| - x \right| - |x| \right| - x?$$

A -2016

B 0

C 2016

D 4032

E 6048

Solution:

Since $x = -2016 < 0$, we have $|x| = 2016$. Thus

$$\begin{aligned} |x| - x &= 2016 - (-2016) \\ &= 4032, \end{aligned}$$

so the innermost absolute value is **4032**. Therefore the full expression is

$$|4032 - 2016| + 2016 = 4032.$$

Thus, the correct answer is **D**.

4. Zoey read 15 books, one at a time. The first book took her 1 day to read, the second book took her 2 days to read, the third book took her 3 days to read, and so on, with each book taking her 1 more day to read than the previous book. Zoey finished the first book on a Monday, and the second on a Wednesday. On what day of the week did she finish her 15th book?

- ☐ A Sunday
- ☒ B Monday
- ☐ C Wednesday
- ☐ D Friday
- ☐ E Saturday

Solution:

The number of days it takes to read 15 books is $\frac{15 \cdot 16}{2} = 120$. Therefore, it is $120 - 1 = 119$ days after the first book is read. This is a multiple of 7, so the 15th book was finished on the same day as the 1st book. Therefore, it was finished on a Monday.

Thus, the correct answer is **B**.

5. The mean age of Amanda's 4 cousins is 8, and their median age is 5. What is the sum of the ages of Amanda's youngest and oldest cousins?

- ☐ A 13
- ☐ B 16
- ☐ C 19
- ☒ D 22
- ☐ E 25

Solution:

The total of the four cousin ages is $4 \cdot 8 = 32$. Since the median is 5, the two middle ages have average 5, so their sum is 10.

Therefore the youngest and oldest ages sum to $32 - 10 = 22$.

Thus, the correct answer is **D**.

6. Laura added two three-digit positive integers. All six digits in these numbers are different. Laura's sum is a three-digit number S . What is the smallest possible value for the sum of the digits of S ?

- A 1
- B 4**
- C 5
- D 15
- E 21

Solution:

To minimize the sum of the addends, use the two smallest nonzero hundreds digits, 1 and 2, and then the four smallest remaining digits, 0, 3, 4, 5. Placing the smaller digits in the tens places shows that every possible sum is at least $104 + 235 = 339$. On the other hand, any three-digit number with digit sum less than 4 is at most 300. Therefore the digit sum of S is at least 4.

The example $157 + 243 = 400$ uses six distinct digits and has digit sum 4. Thus the smallest possible digit sum is 4.

Thus, the correct answer is **B**.

7. The ratio of the measures of two acute angles is $5 : 4$, and the complement of one of these two angles is twice as large as the complement of the other. What is the sum of the degree measures of the two angles?

- A 75
- B 90
- C 135**
- D 150
- E 270

Solution:

Let the smaller angle be x . Then, the larger angle is $\frac{5}{4}x$. Note that their sum is $\frac{9}{4}x$.

Their complements are then $90 - x$ and $90 - \frac{5}{4}x$. Since $90 - x$ is larger than $90 - \frac{5}{4}x$, we know

$$\begin{aligned}90 - x &= 2 \left(90 - \frac{5}{4}x \right) \\90 - x &= 180 - \frac{5}{2}x \\x &= 60.\end{aligned}$$

Therefore, the sum is

$$\frac{9}{4} \cdot 60 = 135.$$

Thus, the correct answer is **C**.

8. What is the tens digit of $2015^{2016} - 2017$?

A 0

B 1

C 3

D 5

E 8

Solution:

To find the tens digit, we first need to find the number modulo 100. First, we can find $2015^{2016} \pmod{100}$. We can do this with the Chinese Remainder Theorem by first getting the number $\pmod{4}$ and then $\pmod{25}$.

The number

$$2015^{2016} \equiv 0 \pmod{25}$$

since it is a multiple of 25.

Then, observe that

$$\begin{aligned} 2015^{2016} &\equiv (-1)^{2016} \\ &\equiv 1 \pmod{4}. \end{aligned}$$

By the Chinese remainder theorem, we can get that our number is congruent to 25 $\pmod{100}$.

Therefore,

$$\begin{aligned} 2015^{2016} - 2017 \\ &\equiv 25 - 17 \pmod{100} \\ &\equiv 8 \pmod{100}. \end{aligned}$$

This means the tens digit is 0.

Thus, the correct answer is **A**.

9. All three vertices of $\triangle ABC$ lie on the parabola defined by $y = x^2$, with A at the origin and $\overline{BC}$ parallel to the x -axis. The area of the triangle is 64. What is the length of BC ?

A 4

B 6

C 8

D 10

E 16

Solution:

Let the points be

$$A = (0, 0),$$

$$B = (x_1, y_1),$$

$$C = (x_2, y_2).$$

Then, since BC is parallel with the x -axis, we know $y_1 = y_2$, which we will let be y . Then, $x_1^2 = y = x_2^2$, so $x_1^2 = x_2^2$. This implies that either $x_1 = -x_2$ or $x_1 = x_2$. The second option cannot happen since that would set two points as the same, which would create an area of 0. As such, let $x = -x_1 = x_2$. Then $y = x^2$ also.

Then, the points are

$$A = (0, 0),$$

$$B = (-x, x^2),$$

$$C = (x, x^2).$$

With a base of BC , the length is $2x$ and the height is x^2 . This would make the area $x^3 = 64$. Therefore, $x = 4$, so $BC = 2 \cdot 4 = 8$.

Thus, the correct answer is **C**.

10. A thin piece of wood of uniform density in the shape of an equilateral triangle with side length 3 inches weighs 12 ounces. A second piece of the same type of wood, with the same thickness, also in the shape of an equilateral triangle, has side length 5 inches. Which of the following is closest to the weight, in ounces, of the second piece?

A 14.0

B 16.0

C 20.0

D 33.3

E 55.6

Solution:

The surface area is increased by a factor of $\frac{5}{3}^2$ since the side lengths are increased by a factor of $\frac{5}{3}$. Also since the thickness is constant, the volume is scaled up by that much. Then, the wood having the same density makes the weight increase by a factor of $\frac{5}{3}^2$ as well, so the new weight is

$$12 \cdot \frac{25}{9} = \frac{100}{3}.$$

This is approximately 33.3.

Thus, the correct answer is **D**.

11. Carl decided to fence in his rectangular garden. He bought 20 fence posts, placed one on each of the four corners, and spaced out the rest evenly along the edges of the garden, leaving exactly 4 yards between neighboring posts. The longer side of his garden, including the corners, has twice as many posts as the shorter side, including the corners. What is the area, in square yards, of Carl's garden?

A 256

B 336

C 384

D 448

E 512

Solution:

Let l be the number of posts on a long side, including corners, and let s be the number of posts on a short side, including corners. The total number of posts is

$$2l + 2s - 4 = 20,$$

where the -4 avoids double-counting the corners.

Since a long side has twice as many posts as a short side, $l = 2s$. Thus $2(2s) + 2s - 4 = 20$, so $s = 4$ and $l = 8$.

There are 3 equal gaps on a short side and 7 equal gaps on a long side. With 4 yards between neighboring posts, the side lengths are 12 and 28, so the area is $12 \cdot 28 = 336$.

Thus, the correct answer is **B**.

12. Two different numbers are selected at random from $\{1, 2, 3, 4, 5\}$ and multiplied together. What is the probability that the product is even?

A 0.2

B 0.4

C 0.5

D 0.7

E 0.8

Solution:

The product is odd if and only if both numbers are odd. There are $\binom{3}{2} = 3$ ways to do this out of a possible $\binom{5}{2} = 10$ ways. This makes the probability of it being odd equal to $\frac{3}{10} = 0.3$. This means the probability it is even is $1 - 0.3 = 0.7$.

Thus, the correct answer is **D**.

13. At Megapolis Hospital one year, multiple-birth statistics were as follows: Sets of twins, triplets, and quadruplets accounted for 1000 of the babies born. There were four times as many sets of triplets as sets of quadruplets, and there were three times as many sets of twins as sets of triplets. How many of these 1000 babies were in sets of quadruplets?

- A 25
- B 40
- C 64
- D 100**
- E 160

Solution:

Let w , r , q be the numbers of sets of twins, triplets, and quadruplets. Then

$$2w + 3r + 4q = 1000.$$

The statement gives $r = 4q$ and $w = 3r = 12q$.

Substituting gives $2(12q) + 3(4q) + 4q = 40q = 1000$, so $q = 25$. The number of babies in sets of quadruplets is $4q = 100$.

Thus, the correct answer is **D**.

14. How many squares whose sides are parallel to the axes and whose vertices have coordinates that are integers lie entirely within the region bounded by the line $y = \pi x$, the line $y = -0.1$ and the line $x = 5.1$?

A 30

B 41

C 45

D 50

E 57

Solution:

A square must lie above $y = -0.1$, to the left of $x = 5.1$, and below $y = \pi x$. Since π is a little more than 3, the lattice heights available at $x = 1, 2, 3, 4$ are 3, 6, 9, 12, and side lengths larger than 3 cannot fit.

Count by side length using the top-left lattice point. For side length 1, there are $3 + 6 + 9 + 12 = 30$ choices. For side length 2, there are $2 + 5 + 8 = 15$ choices. For side length 3, there are $1 + 4 = 5$ choices.

The total is $30 + 15 + 5 = 50$.

Thus, the correct answer is **D**.

15. All the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9 are written in a 3×3 array of squares, one number in each square, in such a way that if two numbers are consecutive then they occupy squares that share an edge. The numbers in the four corners add up to 18. What is the number in the center?

- A 5
- B 6
- C 7**
- D 8
- E 9

Solution:

Color the array like a checkerboard, with the center and four corners one color and the four edge squares the other color. Consecutive numbers occupy adjacent squares, so the path 1, 2, . . . , 9 alternates colors. Therefore all five odd numbers occupy one color class and all four even numbers occupy the other.

The color class consisting of the center and four corners has five squares, so it contains the five odd numbers. Hence the sum of the corners and center is

$$1 + 3 + 5 + 7 + 9 = 25.$$

Since the corners have a sum of 18, the center has a value of $25 - 18 = 7$.

Thus, the correct answer is **C**.

16. The sum of an infinite geometric series is a positive number S , and the second term in the series is 1. What is the smallest possible value of S ?

A $\frac{1 + \sqrt{5}}{2}$

B 2

C $\sqrt{5}$

D 3

E 4

Solution:

Let the first value of the series be a , and let the ratio be r . Thus,

$$\begin{aligned} S &= \frac{a}{1 - r} \\ &= \frac{ar}{r(1 - r)} \\ &= \frac{1}{r(1 - r)}. \end{aligned}$$

This means we have to find r that maximizes

$$r(1 - r) = 0.25 - (r - 0.5)^2.$$

This maximization will happen with $r = 0.5$.

Therefore, $S = \frac{1}{0.5(0.5)} = 4$.

Thus, the correct answer is **E**.

17. All the numbers 2,3,4,5,6,7 are assigned to the six faces of a cube, one number to each face. For each of the eight vertices of the cube, a product of three numbers is computed, where the three numbers are the numbers assigned to the three faces that include that vertex. What is the greatest possible value of the sum of these eight products?

A 312

B 343

C 625

D 729

E 1680

Solution:

Pair opposite faces as (a_1, a_2) , (b_1, b_2) , and (c_1, c_2) . Each vertex product uses one number from each pair, so the sum of all eight vertex products is

$$(a_1 + a_2)(b_1 + b_2)(c_1 + c_2).$$

The six face labels sum to $2 + 3 + 4 + 5 + 6 + 7 = 27$, so the three opposite-pair sums have total 27. Their product is maximized when the sums are as equal as possible, namely 9, 9, 9, giving at most $9^3 = 729$.

This maximum is attainable by pairing 2 with 7, 3 with 6, and 4 with 5. Hence the greatest possible sum is 729.

Thus, the correct answer is **D**.

18. In how many ways can 345 be written as the sum of an increasing sequence of two or more consecutive positive integers?

- A 1
- B 3
- C 5
- D 6
- E 7**

Solution:

Suppose the sequence has length s and first term x . Then

$$345 = s \left(x + \frac{s-1}{2} \right),$$

or

$$s(2x + s - 1) = 690.$$

Thus s must be a divisor of 690, with $x = \frac{\frac{690}{s} - s + 1}{2}$ a positive integer. Checking the possible divisor lengths gives

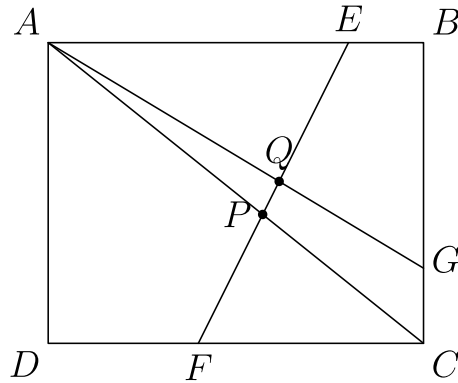
$$s = 2, 3, 5, 6, 10, 15, 23.$$

These are the only lengths that keep x positive and integral.

Therefore there are 7 representations.

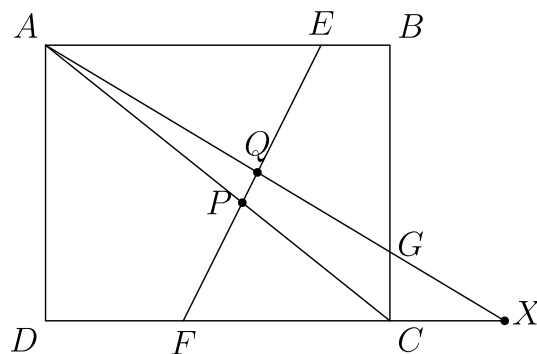
Thus, the correct answer is **E**.

19. Rectangle $ABCD$ has $AB = 5$ and $BC = 4$. Point E lies on $\overline{AB}$ so that $EB = 1$, point G lies on $\overline{BC}$ so that $CG = 1$, and point F lies on $\overline{CD}$ so that $DF = 2$. Segments $\overline{AG}$ and $\overline{AC}$ intersect $\overline{EF}$ at Q and P , respectively. What is the value of $\frac{PQ}{EF}$?



- A $\frac{\sqrt{3}}{16}$
- B $\frac{\sqrt{2}}{13}$
- C $\frac{9}{82}$
- D $\frac{10}{91}$**
- E $\frac{1}{9}$

Solution:



We have $AE = AB - EB = 4$ and $FC = DC - DF = 3$. Since $AE \parallel FC$, the triangles AEP and CFP are similar. Therefore

$$\frac{PF}{PE} = \frac{FC}{AE} = \frac{3}{4},$$

so $\frac{PF}{EF} = \frac{3}{7}$.

Extend AG to meet line CD at X . Because $BG = 3$, similarity gives

$$\frac{AD}{DX} = \frac{BG}{AB}, \quad \frac{4}{DX} = \frac{3}{5},$$

and hence $DX = \frac{20}{3}$. Thus $FX = DX - DF = \frac{14}{3}$. Since $AE \parallel FX$, triangles AEQ and XFQ are similar, so

$$\frac{QF}{QE} = \frac{FX}{AE} = \frac{7}{6}.$$

Consequently $\frac{QF}{EF} = \frac{7}{13}$.

It follows that

$$\begin{aligned} \frac{PQ}{EF} &= \frac{QF}{EF} - \frac{PF}{EF} \\ &= \frac{7}{13} - \frac{3}{7} = \frac{10}{91}. \end{aligned}$$

Thus, the correct answer is **D**.

20. A dilation of the plane—that is, a size transformation with a positive scale factor—sends the circle of radius 2 centered at $A(2, 2)$ to the circle of radius 3 centered at $A'(5, 6)$. What distance does the origin $O(0, 0)$, move under this transformation?

- ☐ A 0
- ☐ B 3
- ☒ C $\sqrt{13}$
- ☐ D 4
- ☐ E 5

Solution:

The dilation scale factor is $\frac{3}{2}$, since the radius changes from 2 to 3. The center of dilation C lies on the line through $A(2, 2)$ and $A'(5, 6)$.

The vector from A to A' is $(3, 4)$. If C is the dilation center, then the scale factor gives

$$A' - A = \frac{1}{2}(A - C).$$

Thus $A - C = 2(3, 4)$, so $C = (2, 2) - 2(3, 4) = (-4, -6)$.

Under a scale factor $\frac{3}{2}$ dilation about C , a point moves by half its distance from C . Since $CO = \sqrt{(-4)^2 + (-6)^2} = \sqrt{52}$, the origin moves $\frac{1}{2}\sqrt{52} = \sqrt{13}$.

Thus, the correct answer is **C**.

21. What is the area of the region enclosed by the graph of the equation $x^2 + y^2 = |x| + |y|$?

A $\pi + \sqrt{2}$

B $\pi + 2$

C $\pi + 2\sqrt{2}$

D $2\pi + \sqrt{2}$

E $2\pi + 2\sqrt{2}$

Solution:

The equation is symmetric in all four quadrants. In the first quadrant it becomes

$$x^2 + y^2 = x + y,$$

or

$$\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{2}.$$

In the first quadrant, the enclosed region is the triangle under $x + y = 1$, with area $\frac{1}{2}$, plus a semicircle of radius $\sqrt{\frac{1}{2}}$, with area $\frac{\pi}{4}$.

Multiplying by 4, the total area is

$$4 \left(\frac{1}{2} + \frac{\pi}{4} \right) = 2 + \pi.$$

Thus, the correct answer is **B**.

22. A set of teams held a round-robin tournament in which every team played every other team exactly once. Every team won 10 games and lost 10 games; there were no ties. How many sets of three teams $\{A, B, C\}$ were there in which A beat B , B beat C , and C beat A ?

A 385

B 665

C 945

D 1140

E 1330

Solution:

The total number of teams is $10 + 10 + 1 = 21$. The total number of sets is therefore $\binom{21}{3} = 1330$.

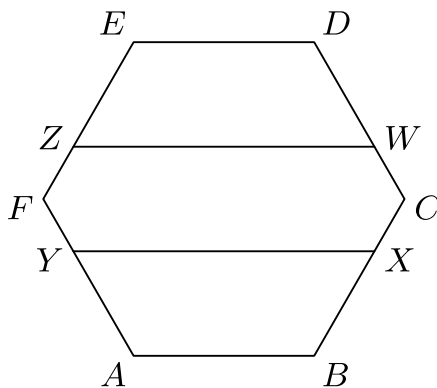
Now, we must subtract the total number of sets such that there is no cycle. This only happens if one team beats the other two teams. There are 21 choices for the team that beat the other two and $\binom{10}{2} = 45$ ways to choose the teams they beat. Thus, the total of non-cycles is $21 \cdot 45 = 945$. This means the total number of cycles is $1330 - 945 = 385$.

Thus, the correct answer is **A**.

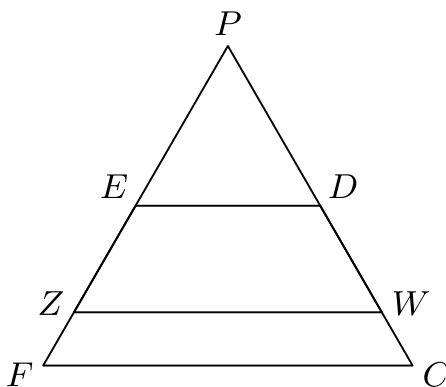
23. In regular hexagon $ABCDEF$, points W , X , Y , and Z are chosen on sides $\overline{BC}$, $\overline{CD}$, $\overline{EF}$, and $\overline{FA}$ respectively, so lines AB , ZW , YX , and ED are parallel and equally spaced. What is the ratio of the area of hexagon $WCXYFZ$ to the area of hexagon $ABCDEF$?

- A $\frac{1}{3}$
- B $\frac{10}{27}$
- C $\frac{11}{27}$**
- D $\frac{4}{9}$
- E $\frac{13}{27}$

Solution:



Extend $\overline{FE}$ and $\overline{CD}$ until they meet at P .



Let d be the distance between ZW and FC . Because the four given lines are equally spaced and FC lies halfway between ZW and YX , the distance from ED to ZW is $2d$. Let the altitude from P to ED be h . The equilateral triangles PED and PFC have side lengths in the ratio $1 : 2$, so their altitudes satisfy

$$h + 3d = 2h,$$

giving $h = 3d$. Therefore the side-length ratio of PZW to PED is

$$\frac{h + 2d}{h} = \frac{5}{3}.$$

Taking $[PED] = 1$, similarity gives $[PZW] = \frac{25}{9}$ and $[PFC] = 4$. Hence

$$\begin{aligned} [ZWCF] &= 4 - \frac{25}{9} = \frac{11}{9}, \\ [EDCF] &= 4 - 1 = 3. \end{aligned}$$

The desired hexagon consists of two congruent copies of $ZWCF$, while the regular hexagon consists of two congruent copies of $EDCF$. Thus the requested ratio is

$$\frac{\frac{11}{9}}{3} = \frac{11}{27}.$$

Thus, the correct answer is **C**.

24. How many four-digit positive integers $abcd$, with $a \neq 0$, have the property that the three two-digit integers $ab < bc < cd$ form an increasing arithmetic sequence?

One such number is 4692, where $a = 4, b = 6, c = 9$, and $d = 2$.

- A 9
- B 15
- C 16
- D 17**
- E 20

Solution:

From $ab < bc < cd$, we have $a \leq b \leq c$. The arithmetic-sequence condition is

$$\begin{aligned} 2(10b + c) &= (10a + b) \\ &\quad + (10c + d), \end{aligned}$$

which rearranges to

$$10(a - 2b + c) = -b + 2c - d.$$

The right side is a multiple of 10. Because $b \leq c$ and all four symbols are digits with $b \geq 1$, it lies between -9 and 17 . Hence it is either 0 or 10.

Case 1 :

$$\begin{aligned} -b + 2c - d &= 10, \\ a &= 2b - c + 1. \end{aligned}$$

We can look at the possible values of c .

$c = 6 : b + d = 2, 2b - 5 = a$. Thus, $b \leq 2$ from the first equation, but can't work for the second equation.

$c = 7 : b + d = 4, 2b - 6 = a$. Thus, $b \leq 4$ from the first equation, and $b > 3$ from the second equation. This makes one case for $b = 4$.

$c = 8 : b + d = 6, 2b - 7 = a$. Thus, $b \leq 6$ from the first equation, and $b > 3$ from the second equation. This makes three cases for $b = 4, 5, 6$.

$c = 9 : b + d = 8, 2b - 8 = a$. Thus, $b \leq 8$ from the first equation, and $b > 4$ from the second equation. This makes four cases for $b = 5, 6, 7, 8$. Altogether this case gives

$$\begin{aligned} &2470, 1482, 3581, 5680, \\ &2593, 4692, 6791, 8890, \end{aligned}$$

for 8 solutions.

Case 2 :

$$\begin{aligned} -b + 2c - d &= 0, \\ 2b - a - c &= 0, \end{aligned}$$

which means the digits are an arithmetic sequence.

If the difference is 1, then $1 \leq a \leq 6$ makes 6 solutions.

If the difference is 2, then $1 \leq a \leq 3$ makes 3 solutions. A difference of at least 3 would force $d = a + 3r > 9$. This case therefore gives 9 solutions.

In total, the number of solutions is $8 + 9 = 17$.

Thus, the correct answer is **D**.

25. Let

$$f(x) = \sum_{k=2}^{10} ([kx] - k[x]),$$

where $\lfloor r \rfloor$ denotes the greatest integer less than or equal to r . How many distinct values does $f(x)$ assume for $x \geq 0$?

- ☒ A 32
- ☐ B 36
- ☐ C 45
- ☐ D 46
- ☐ E infinitely many

Solution:

Write $x = \lfloor x \rfloor + t$, where $0 \leq t < 1$. Then

$$\lfloor kx \rfloor - k\lfloor x \rfloor = \lfloor kt \rfloor,$$

so $f(x)$ depends only on the fractional part t .

The value of f changes only when t crosses a fraction $\frac{i}{k}$, where $2 \leq k \leq 10$ and $1 \leq i < k$. The number of distinct such fractions in $(0, 1)$ is

$$\begin{aligned} & \varphi(2) + \varphi(3) \\ & \quad + \cdots + \varphi(10) \\ & = 1 + 2 + 2 + 4 + 2 \\ & \quad + 6 + 4 + 6 + 4 \\ & = 31. \end{aligned}$$

Including the initial value before the first breakpoint, f assumes $31 + 1 = 32$ distinct values.

Thus, the correct answer is **A**.

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