

2016 AMC 10A Solutions

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1. What is the value of $\frac{11! - 10!}{9!}$?

A 99

B 100

C 110

D 121

E 132

Solution:

We can factor a $9!$ out of the numerator and simplify.

$$\begin{aligned}\frac{9!(11 \cdot 10 - 10)}{9!} &= 11 \cdot 10 - 10 \\ &= 110 - 10 \\ &= 100.\end{aligned}$$

Thus, the correct answer is **B**.

2. For what value of x does

$$10^x \cdot 100^{2x} = 1000^5?$$

- ☐ A 1
- ☐ B 2
- ☒ C 3
- ☐ D 4
- ☐ E 5

Solution:

Rewrite all bases as powers of 10:

$$10^x \cdot 100^{2x} = 10^x \cdot (10^2)^{2x} = 10^{5x},$$

and

$$1000^5 = (10^3)^5 = 10^{15}.$$

Therefore $5x = 15$, so $x = 3$.

Thus, the correct answer is **C**.

3. For every dollar Ben spent on bagels, David spent 25 cents less. Ben paid \$12.50 more than David. How much did they spend in the bagel store together?

A \$37.50

B \$50.00

C \$87.50

D \$90.00

E \$92.50

Solution:

For each dollar Ben spent, David spent 75 cents, so the difference was 25 cents. The total difference \$12.50 is therefore one quarter of Ben's spending.

Ben spent $12.50 \div 0.25 = 50$ dollars, and David spent 37.50 dollars. Together they spent 87.50 dollars.

Thus, the correct answer is **C**.

4. The remainder can be defined for all real numbers x and y with $y \neq 0$ by

$$\text{rem}(x, y) = x - y \left\lfloor \frac{x}{y} \right\rfloor,$$

where $\left\lfloor \frac{x}{y} \right\rfloor$ denotes the greatest integer less than or equal to $\frac{x}{y}$. What is the value of $\text{rem}\left(\frac{3}{8}, -\frac{2}{5}\right)$?

- A $-\frac{3}{8}$
- B $-\frac{1}{40}$**
- C 0
- D $\frac{3}{8}$
- E $\frac{31}{40}$

Solution:

Using the formula, we get

$$\begin{aligned}\text{rem}\left(\frac{3}{8}, -\frac{2}{5}\right) &= \frac{3}{8} \\ &\quad + \frac{2}{5} \left\lfloor \frac{\frac{3}{8}}{-\frac{2}{5}} \right\rfloor \\ &= \frac{3}{8} \\ &\quad + \frac{2}{5} \left\lfloor -\frac{15}{16} \right\rfloor \\ &= \frac{3}{8} + \frac{2}{5} \cdot -1 \\ &= \frac{3}{8} - \frac{2}{5} \\ &= -\frac{1}{40}\end{aligned}$$

Thus, the correct answer is **B**.

5. A rectangular box has integer side lengths in the ratio $1 : 3 : 4$. Which of the following could be the volume of the box?

- ☐ A 48
- ☐ B 56
- ☐ C 64
- ☒ D 96
- ☐ E 144

Solution:

Let s be the side length of the smallest side. Then the other two sides are $3s$ and $4s$.

The volume is therefore

$$s \cdot 3s \cdot 4s = 12s^3.$$

Testing out values of s , we see that if $s = 2$, then $12s^3 = 96$, which is an answer choice.

Thus, the correct answer is **D**.

6. Ximena lists the whole numbers 1 through 30 once. Emilio copies Ximena's numbers, replacing each occurrence of the digit 2 by the digit 1. Ximena adds her numbers and Emilio adds his numbers. How much larger is Ximena's sum than Emilio's?

A 13

B 26

C 102

D 103

E 110

Solution:

When Emilio changes a units digit 2 to 1, his copied number is smaller by 1. This happens in 2, 12, 22, for a loss of 3.

When he changes a tens digit 2 to 1, his copied number is smaller by 10. This happens for 20, 21, . . . , 29, for a loss of 100.

Thus Emilio's sum is 103 less than Ximena's sum, so Ximena's sum is 103 larger.

Thus, the correct answer is **D**.

7. The mean, median, and mode of the 7 data values

60, 100, x , 40, 50, 200, 90

are all equal to x . What is the value of x ?

A 50

B 60

C 75

D 90

E 100

Solution:

The sum of the seven data values is $540 + x$. If the mean equals x , then

$$\frac{540 + x}{7} = x,$$

so $540 = 6x$ and $x = 90$.

With $x = 90$, the ordered data set is 40, 50, 60, 90, 90, 100, 200. Its median is 90, and its mode is also 90, so the value works.

Thus, the correct answer is **D**.

8. Trickster Rabbit agrees with Foolish Fox to double Fox's money every time Fox crosses the bridge by Rabbit's house, as long as Fox pays 40 coins in toll to Rabbit after each crossing. The payment is made after the doubling. Fox is excited about his good fortune until he discovers that all his money is gone after crossing the bridge three times. How many coins did Fox have at the beginning?

- A 20
- B 30
- C 35**
- D 40
- E 45

Solution:

We know that Fox has 0 coins at the end. Then before paying the final toll, Fox had 40 coins.

Then he had $40 \div 2 = 20$ coins before the doubling. Then before paying the toll for the second crossing, he had $20 + 40 = 60$ coins.

Before the doubling on the second crossing, he had $60 \div 2 = 30$ coins. On the first crossing before the toll, Fox had $30 + 40 = 70$ coins.

Finally, before the first doubling, Fox had $70 \div 2 = 35$ coins.

Thus, the correct answer is **C**.

9. A triangular array of 2016 coins has 1 coin in the first row, 2 coins in the second row, 3 coins in the third row, and so on up to N coins in the N th row. What is the sum of the digits of N ?

- A 6
- B 7
- C 8
- D 9**
- E 10

Solution:

Recall that the sum of the first N numbers is $\frac{N(N+1)}{2}$.

We want to find N such that

$$\frac{N(N+1)}{2} = 2016.$$

Cross-multiplying and simplifying gives us

$$N^2 + N - 4032 = 0.$$

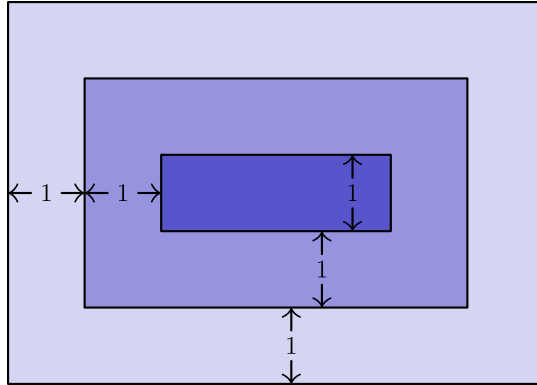
Factoring gives us

$$(N - 63)(N + 64) = 0.$$

We want the positive value so $N = 63$. Adding together the digits gives us 9.

Thus, the correct answer is **D**.

10. A rug is made with three different colors as shown. The areas of the three differently colored regions form an arithmetic progression. The inner rectangle is one foot wide, and each of the two shaded regions is 1 foot wide on all four sides. What is the length in feet of the inner rectangle?



- A 1
- B 2**
- C 4
- D 6
- E 8

Solution:

Let l be the length of the inner rectangle. Then the area of the inner rectangle is l .

The area of the middle region is going to be

$$(l + 2) \cdot 3 - l = 2l + 6.$$

The area of the outer region is

$$(l + 4) \cdot 5 - (l + 2) \cdot 3 = 2l + 14.$$

We know that these 3 values form an arithmetic sequence. That means that

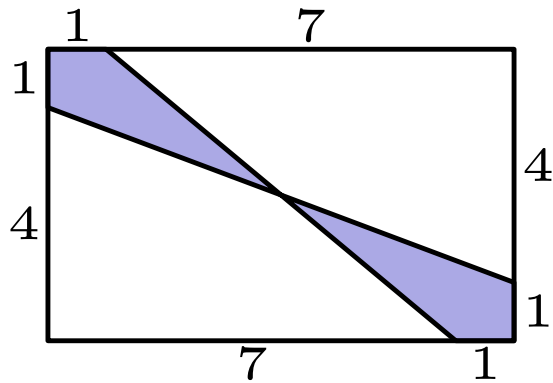
$$l + 2l + 14 = 2(2l + 6)$$

$$3l + 14 = 4l + 12$$

$$l = 2$$

Thus, the correct answer is **B**.

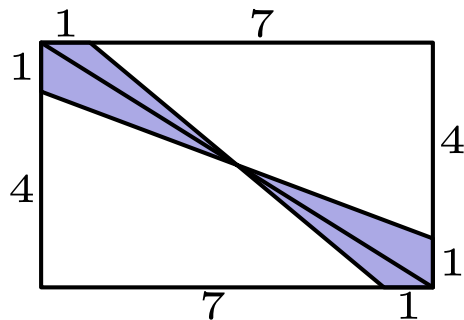
11. What is the area of the shaded region of the given 8×5 rectangle?



- A $4\frac{3}{5}$
- B 5
- C $5\frac{1}{4}$
- D $6\frac{1}{2}$**
- E 8

Solution:

We can split the region into 4 triangles with bases of 1.



All four triangles have base 1. Two have height $8 \div 2 = 4$, and the other two have height $5 \div 2 = \frac{5}{2}$.

The sum of the areas of the triangles is

$$2 \cdot \frac{1}{2} \left(1 \cdot \frac{5}{2} \right) + 2 \cdot \frac{1}{2} (1 \cdot 4) = 6\frac{1}{2}$$

Thus, the correct answer is **D**.

12. Three distinct integers are selected at random between 1 and 2016, inclusive. Which of the following is a correct statement about the probability p that the product of the three integers is odd?

☒ A $p < \frac{1}{8}$

☐ B $p = \frac{1}{8}$

☐ C $\frac{1}{8} < p < \frac{1}{3}$

☐ D $p = \frac{1}{3}$

☐ E $p > \frac{1}{3}$

Solution:

The product is odd exactly when all three selected integers are odd. There are 1008 odd and 1008 even integers from 1 to 2016.

Because the integers are selected without replacement,

$$p = \frac{1008}{2016} \cdot \frac{1007}{2015} \cdot \frac{1006}{2014}.$$

The first factor is $\frac{1}{2}$, and each of the next two factors is slightly less than $\frac{1}{2}$. Therefore $p < \frac{1}{8}$.

Thus, the correct answer is **A**.

13. Five friends sat in a movie theater in a row containing 5 seats, numbered 1 to 5 from left to right. (The directions “left” and “right” are from the point of view of the people as they sit in the seats.)

During the movie Ada went to the lobby to get some popcorn. When she returned, she found that Bea had moved two seats to the right, Ceci had moved one seat to the left, and Dee and Edie had switched seats, leaving an end seat for Ada. In which seat had Ada been sitting before she got up?

- ☐ A 1
- ☒ B 2
- ☐ C 3
- ☐ D 4
- ☐ E 5

Solution:

The sum of the occupied seat numbers is $1 + 2 + 3 + 4 + 5 = 15$ both before and after the moves. Dee and Edie merely switch seats, so their combined displacement is 0. Bea’s displacement is $+2$, and Ceci’s is -1 . Therefore Ada’s displacement must be -1 so that the total displacement is 0.

Ada returns to an end seat. Since she moves one seat left, she must move from seat 2 to seat 1.

Thus, the correct answer is **B**.

14. How many ways are there to write 2016 as the sum of twos and threes, ignoring order?
(For example, $1008 \cdot 2 + 0 \cdot 3$ and $402 \cdot 2 + 404 \cdot 3$ are two such ways.)

A 236

B 336

C 337

D 403

E 672

Solution:

The problem can be rewritten as an equation $2x + 3y = 2016$, where x is the number of twos and y is the number of threes.

The goal is to find the number of multiples of 3 that can be subtracted from 2016 to result in an even number.

This can be achieved by the pairs of $(1008, 0)$ up to $(0, 672)$ with y being incremented by 2.

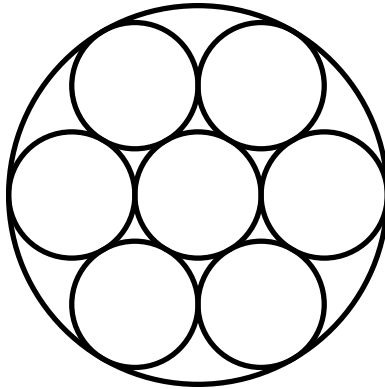
This gives us

$$\frac{672}{2} + 1 = 337$$

solutions for y and x .

Thus, the correct answer is **C**.

15. Seven cookies of radius 1 inch are cut from a circle of cookie dough, as shown. Neighboring cookies are tangent, and all except the center cookie are tangent to the edge of the dough. The leftover scrap is reshaped to form another cookie of the same thickness. What is the radius in inches of the scrap cookie?



- ☒ A $\sqrt{2}$
- ☐ B 1.5
- ☐ C $\sqrt{\pi}$
- ☐ D $\sqrt{2\pi}$
- ☐ E π

Solution:

The circle of cookie dough has a radius of 3 inches since it is the same as the diameter plus the radius of a cookie.

The area of the cookie dough is $3^2\pi = 9\pi$, and the cookies have an area of $7 \cdot 1^2\pi = 7\pi$.

The area of the leftover scrap is therefore $9\pi - 7\pi = 2\pi$. This means that its radius is $\sqrt{2}$.

Thus, the correct answer is **A**.

16. A triangle with vertices $A(0, 2)$, $B(-3, 2)$, and $C(-3, 0)$ is reflected about the x -axis, then the image $\triangle A'B'C'$ is rotated counterclockwise about the origin by 90° to produce $\triangle A''B''C''$. Which of the following transformations will return $\triangle A''B''C''$ to $\triangle ABC$?

A counterclockwise rotation about the origin by 90°

B clockwise rotation about the origin by 90°

C reflection about the x -axis

D reflection about the line $y = x$

E reflection about the y -axis

Solution:

To figure out how to reverse the transformations, we can analyze a single point and see what happens to it.

Let (x, y) be the point. After being reflected about the x -axis, the point would go to $(x, -y)$.

Rotating this counterclockwise would put it at (y, x) . The only transformation that puts this back at (x, y) is reflection about $y = x$.

Thus, the correct answer is **D**.

17. Let N be a positive multiple of 5. One red ball and N green balls are arranged in a line in random order. Let $P(N)$ be the probability that at least $\frac{3}{5}$ of the green balls are on the same side of the red ball. Observe that $P(5) = 1$ and that $P(N)$ approaches $\frac{4}{5}$ as N grows large. What is the sum of the digits of the least value of N such that $P(N) < \frac{321}{400}$?

A 12

B 14

C 16

D 18

E 20

Solution:

Think of first arranging the N green balls, then placing the red ball in one of the $N + 1$ gaps. If k green balls are to the left of the red ball, then $N - k$ are to its right.

At least $\frac{3}{5}N$ green balls are on one side exactly when $k \leq \frac{2}{5}N$ or $k \geq \frac{3}{5}N$. Thus the bad gaps are

$$\frac{2}{5}N + 1, \frac{2}{5}N + 2, \dots, \frac{3}{5}N - 1,$$

a total of $\frac{1}{5}N - 1$ gaps.

Therefore

$$\begin{aligned} P(N) &= 1 - \frac{\frac{N}{5} - 1}{N + 1} \\ &= \frac{4N + 10}{5N + 5}. \end{aligned}$$

Solving $\frac{4N+10}{5N+5} < \frac{321}{400}$ gives $N > 479$. The least positive multiple of 5 is 480, whose digit sum is 12.

Thus, the correct answer is **A**.

18. Each vertex of a cube is to be labeled with an integer 1 through 8, with each integer being used once, in such a way that the sum of the four numbers on the vertices of a face is the same for each face. Arrangements that can be obtained from each other through rotations of the cube are considered to be the same. How many different arrangements are possible?

- A 1
- B 3
- C 6**
- D 12
- E 24

Solution:

Opposite faces together use all eight labels, whose sum is 36, so every face must have sum 18.

Put label 1 at one vertex, and let the three adjacent labels be a, b, c . The three vertices adjacent to those across faces are then forced to be

$$\begin{aligned} 17 - a - b, \\ 17 - a - c, \\ 17 - b - c, \end{aligned}$$

and the opposite vertex is $a + b + c - 16$.

Assume the three neighbor labels are listed in increasing order. Substituting triples from 2, 3, ..., 8 into the four forced expressions above gives exactly

$\{a, b, c\}$	four forced labels
$\{4, 6, 8\}$	$\{7, 5, 3, 2\}$
$\{4, 7, 8\}$	$\{6, 5, 2, 3\}$
$\{6, 7, 8\}$	$\{4, 3, 2, 5\}$

For each neighbor set, the six orders around the vertex fall into two groups of three under the 120° rotations about the diagonal through that vertex. Thus each set gives two non-rotationally equivalent arrangements, for $3 \cdot 2 = 6$ arrangements in all.

Thus, the correct answer is **C**.

19. In rectangle $ABCD$, $AB = 6$ and $BC = 3$. Point E between B and C , and point F between E and C are such that $BE = EF = FC$. Segments $\overline{AE}$ and $\overline{AF}$ intersect $\overline{BD}$ at P and Q , respectively.

The ratio $BP : PQ : QD$ can be written as $r : s : t$ where the greatest common factor of r , s , and t is 1. What is $r + s + t$?

- ☐ A 7
- ☐ B 9
- ☐ C 12
- ☐ D 15
- ☒ E 20

Solution:

Since $BC = 3$ and $BE = EF = FC$, we have $BE = 1$ and $BF = 2$. Also $AD = 3$.

From $\triangle APD \sim \triangle EPB$,

$$\frac{BP}{BD} = \frac{BE}{AD + BE} = \frac{1}{4}.$$

From $\triangle AQD \sim \triangle FQB$,

$$\frac{BQ}{BD} = \frac{BF}{AD + BF} = \frac{2}{5}.$$

Therefore $BP = \frac{1}{4}BD$, $PQ = \left(\frac{2}{5} - \frac{1}{4}\right)BD = \frac{3}{20}BD$, and $QD = \frac{3}{5}BD$. Thus

$$\begin{aligned} BP : PQ : QD &= \frac{1}{4} : \frac{3}{20} : \frac{3}{5} \\ &= 5 : 3 : 12. \end{aligned}$$

The sum is $5 + 3 + 12 = 20$.

Thus, the correct answer is **E**.

20. For some particular value of N , when $(a + b + c + d + 1)^N$ is expanded and like terms are combined, the resulting expression contains exactly 1001 terms that include all four variables a, b, c , and d , each to some positive power. What is N ?

A 9

B 14

C 16

D 17

E 19

Solution:

A term that includes all four variables has form $a^v b^w c^x d^y$, with $v, w, x, y \geq 1$, multiplied by some power of 1. If that power is z , then

$$v + w + x + y + z = N.$$

After subtracting 1 from each of v, w, x, y , this becomes a nonnegative stars-and-bars count with 5 variables summing to $N - 4$. The number of such terms is

$$\binom{N}{4}.$$

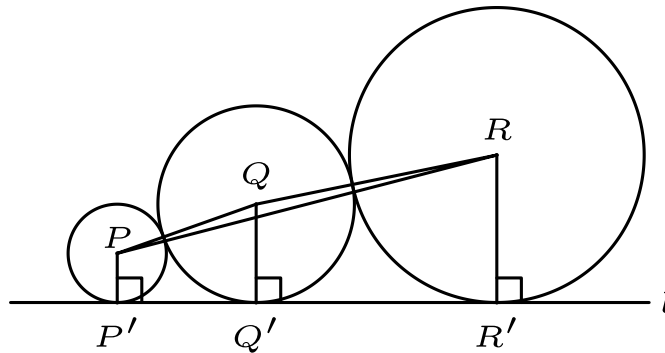
Since $\binom{14}{4} = 1001$, we get $N = 14$.

Thus, the correct answer is **B**.

21. Circles with centers P , Q and R , having radii 1, 2 and 3, respectively, lie on the same side of line l and are tangent to l at P' , Q' and R' , respectively, with Q' between P' and R' . The circle with center Q is externally tangent to each of the other two circles. What is the area of $\triangle PQR$?

- A 0
- B $\sqrt{\frac{2}{3}}$
- C 1
- D $\sqrt{6} - \sqrt{2}$
- E $\sqrt{\frac{3}{2}}$

Solution:



Put the tangent line on the x -axis and take $P = (0, 1)$. Because $PQ = 1 + 2 = 3$ and the centers differ in height by 1, the horizontal distance from P to Q is

$$\sqrt{3^2 - 1^2} = 2\sqrt{2}.$$

Similarly, $QR = 2 + 3 = 5$ and its centers also differ in height by 1, so their horizontal distance is $2\sqrt{6}$.

Thus we may use

$$\begin{aligned} P &= (0, 1), & Q &= (2\sqrt{2}, 2), \\ R &= (2\sqrt{2} + 2\sqrt{6}, 3). \end{aligned}$$

The coordinate-area formula gives

$$\begin{aligned}[PQR] &= \frac{1}{2} \left| 4\sqrt{2} - (2\sqrt{2} + 2\sqrt{6}) \right| \\ &= \sqrt{6} - \sqrt{2}.\end{aligned}$$

Thus, the correct answer is **D**.

22. For some positive integer n , the number $110n^3$ has 110 positive integer divisors, including 1 and the number $110n^3$. How many positive integer divisors does the number $81n^4$ have?

- ☐ A 110
- ☐ B 191
- ☐ C 261
- ☒ D 325
- ☐ E 425

Solution:

The prime factorization is $110 = 2 \cdot 5 \cdot 11$. If $110n^3$ has $110 = 2 \cdot 5 \cdot 11$ divisors, then it has exactly three prime factors, with exponents 1, 4, 10 in some order.

For each of the primes 2, 5, 11, its exponent in $110n^3$ is 1 more than a multiple of 3. The exponents 1, 4, 10 all have this form, so the corresponding exponents in n are 0, 1, 3 in some order.

In $81n^4 = 3^4n^4$, the exponents from n^4 are therefore 0, 4, 12, in some order, along with the exponent 4 on prime 3. Hence the divisor count is

$$\begin{aligned}(0 + 1) \cdot (4 + 1) \cdot (12 + 1) \\ \cdot (4 + 1) = 325.\end{aligned}$$

Thus, the correct answer is **D**.

23. A binary operation $\diamond$ has the properties that

$$a \diamond (b \diamond c) = (a \diamond b) \cdot c$$

and that $a \diamond a = 1$ for all nonzero real numbers a , b , and c . (Here $\cdot$ represents multiplication). The solution to the equation

$$2016 \diamond (6 \diamond x) = 100$$

can be written as $\frac{p}{q}$, where p and q are relatively prime positive integers. What is $p + q$?

A 109

B 201

C 301

D 3049

E 33,601

Solution:

Since $a \diamond a = 1$, substituting $b = c$ in

$$a \diamond (b \diamond c) = (a \diamond b)c$$

gives $a \diamond 1 = (a \diamond b)b$. Also, using $a \diamond (a \diamond a) = (a \diamond a)a$ gives $a \diamond 1 = a$. Therefore $a \diamond b = \frac{a}{b}$.

The equation becomes

$$\begin{aligned} 2016 \diamond (6 \diamond x) &= 2016 \diamond \frac{6}{x} \\ &= \frac{2016}{\frac{6}{x}} \\ &= 336x \\ &= 100. \end{aligned}$$

Thus $x = \frac{25}{84}$, so $p + q = 25 + 84 = 109$.

Thus, the correct answer is **A**.

24. A quadrilateral is inscribed in a circle of radius $200\sqrt{2}$. Three of the sides of this quadrilateral have length 200. What is the length of the fourth side?

A 200

B $200\sqrt{2}$

C $200\sqrt{3}$

D $300\sqrt{2}$

E 500

Solution:

Label the quadrilateral $ABCD$, with $AB = BC = CD = 200$. Let each of these equal chords subtend central angle 2θ . The chord formula gives

$$200 = 2(200\sqrt{2}) \sin \theta,$$

so $\sin \theta = \frac{1}{2\sqrt{2}}$. In particular, $\theta < 30^\circ$, so the smaller central angle subtended by AD is $6\theta < 180^\circ$.

Therefore another use of the chord formula gives

$$AD = 2(200\sqrt{2}) \sin(3\theta).$$

Using $\sin(3\theta) = 3 \sin \theta - 4 \sin^3 \theta$, we obtain

$$\begin{aligned} \sin(3\theta) &= \frac{3}{2\sqrt{2}} - \frac{1}{4\sqrt{2}} \\ &= \frac{5}{4\sqrt{2}}. \end{aligned}$$

Hence

$$AD = 400\sqrt{2} \cdot \frac{5}{4\sqrt{2}} = 500.$$

Thus, the correct answer is **E**.

25. How many ordered triples (x, y, z) of positive integers satisfy

$$\text{lcm}(x, y) = 72,$$

$$\text{lcm}(x, z) = 600,$$

and

$$\text{lcm}(y, z) = 900?$$

A 15

B 16

C 24

D 27

E 64

Solution:

Factor the given least common multiples:

$$72 = 2^3 \cdot 3^2,$$

$$600 = 2^3 \cdot 3 \cdot 5^2,$$

$$900 = 2^2 \cdot 3^2 \cdot 5^2.$$

Since $\text{lcm}(x, y)$ has no factor of 5, neither x nor y has a factor of 5, and z must contain 5^2 .

Write $x = 2^a 3^b$, $y = 2^c 3^d$, and $z = 2^e 3^f 5^2$. The power of 3 in $\text{lcm}(y, z)$ forces $d = 2$, and the power of 2 in $\text{lcm}(x, z)$ forces $a = 3$.

The remaining independent conditions are $\max(b, f) = 1$ and $\max(c, e) = 2$. These have 3 and 5 ordered choices, respectively, so there are $3 \cdot 5 = 15$ triples.

Thus, the correct answer is **A**.

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